Black Hole Thermodynamics

Narit Pidokrajt

Master's Thesis in
Theoretical Physics

Stockholm University
Department of Physics
2003

Abstract

Thermodynamics of various black holes is studied by means of thermodynamic Riemannian curvatures . The thermodynamic Riemannian curvature is a scalar curvature of the Ruppeiner metric, which is defined as a metric tensor on the thermodynamic parameter space, mathematically it is the matrix of second derivatives (Hessian matrix) of the entropy with respect to the energy (mass of the black hole) and other extensive thermodynamic parameters such as the black hole's spin and the electric charge. In ordinary thermodynamics the Ruppeiner metric is a flat metric if the underlying statistical mechanical system is non-interacting such as that of the ideal gas. Furthermore, the curvature scalar of the Ruppeiner metric diverges at the critical point. In this thesis, various black hole families are investigated, i.e., the BTZ, the Reissner-Nordström (RN), the Reissner-Nordström-anti-de Sitter (RNadS), the Kerr and the Kerr-Newman (KN) black holes respectively and it is found that the BTZ and the RN black holes both have vanishing curvatures. The RNadS black hole has a curved geometry and gives interpretable results—namely that its curvature diverges in the extremal limit and its metric changes signature where the thermodynamical stability properties of the black hole change. The Kerr black hole has a non-zero curvature scalar which diverges at the extremal limit, whereas the Ruppeiner metric for KN black hole has a non-trivial curvature and does not give rise any further conclusive remarks.

Contents

List of Figures

List of Tables

Preface

It is no exaggeration to say that there is no other way—unless one wishes something non-scientific—to learn about "black holes" than using our existing and yet-to-be-discovered physics knowledge. By "our existing and yet-to-be discovered physics knowledge" I mean certain knowledge of physics that is capable of telling us certain truths about black holes in a systematically reasonable way, and we do believe that physics is doing a good job. As the author of this thesis, I believe that black holes do exist in the Universe and the reason for that is because there have been certain verifications that followed feasible logics both theoretically and experimentally. I do also believe that there are very many things that we do not know about them owing to the fact that the nature of the black holes are not of the common sense that exists in our mind. As a matter of fact no one has ever seen the black hole with his/her naked eyes and will never! Therefore the best tools that we do have for the time being to study these strange objects (entities) seem to be:

- Physics together with mathematics: So far they both have been serving as theoretical tools to get insights into the nature of black holes to some extent.
- Our imaginations and our dreams: Without dreams humans would not have been flying—the great work by great physicists have always been inspired by their imaginations and dreams. These will be some of the key factors if one aims to study the black holes seriously.
- Modern astronomy and space technologies: With the Hubble space telescope being in orbit we are in hope that it will reveal to us more and better pictures of the Universe, including some indirect measurement of what are believed to be black holes. Recall that this was not possible some 15 years ago, therefore it can be stressed that there is still so much one can learn in the future owing to the speedy advancement of the modern astronomy and space technologies.

Nevertheless, progress in the physics of black holes—either small steps or giants leaps—is yet to come and one cannot say anything more congretely than this. The mysteries of the black holes may prevail for decades but these queer objects will still be of interest to the world of physics inarguably. For the sake of history, I would like to present the timeline for the physics of black holes (black hole physics) and I hope that this will be useful for some of the readers who may not know how the concept (knowledge/physics) of black holes began and how it has developing. This timeline chart was done by Niel Brandt—interestingly under the condition that the "Copyright Notice" is posted as it is on his website I am permitted to copy and paste it here. Thankfully, this text may also be modified up to one's need and my modification will be in *slanted* letters.

1784 : John Michell discusses classical bodies which have escape velocities greater than the speed of light
1795 : Pierre Laplace discusses classical bodies which have escape velocities greater than the speed of light
1916 : Karl Schwarzschild solves the Einstein vacuum field equations for uncharged spherically symmetric systems
1918 : H. Reissner and G. Nordström solve the Einstein-Maxwell field equations for charged spherically symmetric systems
1923 : George Birkhoff proves that the Schwarzschild spacetime geometry is the unique spherically symmetric solution of the Einstein vacuum field equations
1939 : Robert Oppenheimer and Hartland Snyder calculate the collapse of a pressure-free homogeneous fluid sphere and find that it cuts itself off from communication with the rest of the universe
1963 : Roy Kerr solves the Einstein vacuum field equations for uncharged rotating systems
1964 : Roger Penrose proves that an imploding star will necessarily produce a singularity once it has formed an apparent horizon (trapped surface)
1965 : Ezra Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and Robert Torrence solve the Einstein-Maxwell field equations for charged rotating systems
1968 : Brandon Carter uses Hamilton-Jacobi theory to derive first-order equations of motion for a charged particle moving in the external fields of a Kerr-Newman black hole
1969 : Roger Penrose discusses the Penrose process for the extraction of the spin energy from a Kerr black hole
1969 : Roger Penrose proposes the cosmic censorship hypothesis
1971 : Identification of Cygnus X-1/HDE 226868 as a binary black hole candidate system
1972 : Stephen Hawking proves that the area of a classical black hole's event horizon cannot decrease
1972 : James Bardeen, Brandon Carter, and Stephen Hawking propose four laws of black hole mechanics in analogy with the laws of thermodynamics
1972 : Jacob Bekenstein suggests that black holes have an entropy proportional to their surface area due to information loss effects
1974 : Stephen Hawking applies quantum field theory to black hole spacetimes and shows that black holes will radiate particles with a blackbody spectrum which can cause black hole evaporation
1989 : Identification of GS2023+338/V404 Cygni as a binary black hole candidate system
1992 : The 2+1 dimensional black hole was discovered by Banãdos, Teitelboim and Zanelli—known as the BTZ black hole. It is merely a theoretical playground for physicists but proved to be valuable as a platform for string theory and suchs.
1994 : Hubble Space Telescope (HST) confirms existence of massive black hole at heart of active galaxy M87, located 50 million light-years from Earth in the constellation Virgo.

What this thesis is concerned with is the *black hole thermodynamics* as it is entitled—the black holes are studied thermodynamically by means of thermodynamic (classical) fluctuation theory based on the concept of the Riemannian geometry. Contained in this volume are five chapters, two appendices and an index. The first two chapters are basically introducing messages especially for those who might not be familiar with the subject of general relativity. The third chapter is basically the main stream of this work, which is an application of the the thermodynamic fluctuation theory to the black holes. Results will be given in chapter four followed by conclusions and speculations of the results. The two appendices are brief information on ordinary thermodynamics and curvature scalars respectively. As a matter of fact, I am fully aware that writing a thesis is not a so easy task that can be accomplished in a few months' time, but with the limited amount of time of roughly 5 months I am pleased enough to present this work as it is in your hand.

It would be an out-of-date fashion to have this thesis published only on papers, therefore this thesis is also made available on the world-wide web in two formats as follows:

`www.physto.se/~narit/bh.pdf` (Acrobat reader program is required)

or

`www.physto.se/~narit/bh.ps` (In Postscript format, Ghostview program may be needed)

I openly welcome comments or suggestions in any aspect that the reader may have pertaining to this work. I can be reached easily via email at `narit@physto.se` and finally—enjoy the reading!

Narit Pidokrajt
Stockholm, April 2003

Acknowledgments

There would not be this thesis had I not met Professor Ingemar Bengtsson—my supervisor—who has been a significantly constant source of encouragement and knowledge for me since the first day I arrived at Fysikum in Stockholm. Each discussion with him has always been fruitful and essential for me to achieve the goal of this project. I would also like to thank him a lot for being critical in proofreading this thesis. I am grateful to Dr. Jan Åman for his collaboration and for many interesting conversations. Many thanks go to Åsa Ericsson for exchanging thoughts with me on various things both inside and outside physics. I wish to thank Lars Samuelsson for introducing me to the GRtensor program which makes life a bit easier. (Thanks go to those in Canada who created it as well!) Dr. Sayed Fawad Hassan is acknowledged for teaching me how to create transparencies with LaTeX. I am thankful to Moundheur Zarroug and Professor Alexander Zheltukhin (my office-mates since the end of February 2003) for widening my views on France and Ukraine plus some stories about the Russian physicists that I have never heard of from anyone else. Every member of the Fält-och-partikel (FOP) Group is thankworthy for making this department a very attractive and stimulating work place.

I am indebted to Isabella Malmnäs for helping me to get a little home in Katrineholm as it is always hard to get a place to live in Stockholm, and for making my life besides physics an emotional and delightful life. Lastly and unforgettably I wish to thank my family for every support that they have been providing to me although they are so far way.

Chapter 1

Introduction

> *Time and space are modes in which we think and not conditions in which we live in.*
>
> –A. Einstein

Black holes are some of the most exotic[1] entities encountered in physics of the present time[2]. The nature of spacetime within a black hole is enough to make the science of black holes seem more like a science fiction. Even more astounding are the connections of black hole physics with thermodynamics. Classically we would expect the black hole to be a perfectly dead star, namely it should have an absolute zero as a physical temperature. But it was not so since Hawking has found a startling discovery that the black hole radiates thermally [15, 16] whereas Bekenstein suggested that there is an entropy associated with the black hole [5], i.e., the *black hole entropy*. However, that the black hole has an entropy first arose from the realization that its event horizon surface area exhibits remarkable tendency to increase when undergoing any transformation as noticed by Floyd and Penrose [27] and later supported by Christodoulou [10]. Hawking [14] was the first to give a general proof that the surface area of the black hole cannot decrease in any process and additionally he showed that when two black holes coalesce, the area of the resulting black hole cannot be smaller than the sum of the initial areas. Easily speaking, it is clear that changes in black hole generally occur in the direction of increasing area. This reminds us of the second law of ordinary thermodynamics which states that changes of a closed thermodynamic system take place in the direction of increasing entropy. Historically, physicists were not convinced about the validity the black hole thermodynamics before Hawking radiation was discovered.

Despite the written-down laws of black hole thermodynamics we have never been able to do any real experiment to verify them, all we can do is the *gedanken experiment*[3]. It is thus legitimate to say we "kind of" perform gedanken experiments on the black holes—thermodynamically—by means of thermodynamic fluctuation theory using the language of Riemannian geometry in this thesis work. By the

[1] although one may take their existence for granted!

[2] Our conclusions concerning black hole existence are based on observations of matter moving in the vicinity of the black holes. The evidence may well improve when gravitational waves are detected.

[3] which means thought experiment. It is simply an experiment that can be done in principle, and which is useful to think about in order to clarify one's ideas.

virtue of the thermodynamic Riemannian curvature (i.e., the curvature scalar[4]) of the Ruppeiner metric of the black hole under consideration (to be discussed in Sect. 3.3.1)—we hopefully will obtain some new information on the black holes. We investigate also the Weinhold metric for each black hole (Sect. 3.3.2), which is the metric conformally related to the Ruppeiner metric, and it turns that it gives us a helping hand when the Ruppeiner metric is too complicated to deal with.

Throughout this report, we will set $G = \hbar = c = k_{\mathrm{B}} = 1$ unless otherwise stated. These units are sometime referred to as *geometrized units*.

[4]The curvature scalar is independent of the coordinate system in which it is calculated. This point is essential when we want to endow the curvature with an intrinsic physical meaning.

Chapter 2

Black Holes and General Relativity

> *The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time.*
>
> –S. Chandrashekar

Black holes[1] were at first only a speculation as a result of calculation by Laplace in 1795[2] when he discussed the classical body with escape velocity greater than that of light, but his idea did not attract much attention. Later in 1916 Karl Schwarzschild was able solve the Einstein field equation in the vacuum for uncharged spherical systems and his solution is known as the *Schwarzschild solution*, which implies the simplest black hole type namely the *Schwarzschild black hole* that is only determined by a single parameter, namely its mass M.

The physics of black holes is largely based on Einstein's general theory of relativity (GR), which is a theory of gravitation. Relativity is a geometrical theory because the mathematical study of spacetime, whether curved or flat, is geometry[3]. There is no doubt that GR (including SR as well since SR is just a special solution of GR) is one of the most successful physical theories of mankind (it of course cannot tell everything about the universe[4],) owing to its predictions that have been confirmed by a number of experimental tests.[5]

In GR physics is expressed in terms of tensors. Each kind of spacetime is assigned with the proper time and space interval which are described by the *line element* or *metric* which is an invariant interval. For example, in SR there is a *Minkowski line element* which takes the form

[1] Alternative phrases for the black holes used in the pre-1969 literature were "frozen stars" and "collapsed stars" see [23].

[2] At that time the speed of light could not be precisely measured and there was no concept of the speed of light as the ultimate speed in the Universe. Einstein's special theory of relativity (SR) was not publicly known until 1905.

[3] It can simply be said that in SR we deal with the Euclidean geometry, which is a geometry of flat spacetime. Whereas in GR we deal with the non-Euclidean geometry which is the Riemannian spacetime.

[4] There have been attempts to create a Theory Of Everything (TOE), but it is still far from an ultimate success due to the lack of knowledge of gravity at the quantum level.

[5] See [39] for example.

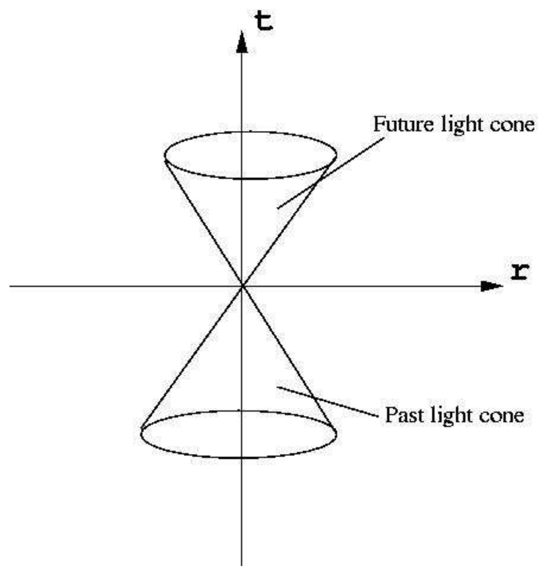

Figure 2.1: Light cones in Minkowski spacetime. Any point in the future light cone $r = t$ can be reached by a particle or signal with speed less than c.

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.1}$$

or in a tensorial form

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.2}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \text{diag}(-1, 1, 1, 1) \tag{2.3}$$

and $\eta_{\mu\nu}$ is called the *Minkowski metric tensor* or just the *Minkowski metric* for this particular Cartesian coordinates. This metric can turn out to be a positive, negative and zero quantity—it corresponds to a space-like, time-like and light-like metric interval repectively. If we use a general coordinate system[6],

$$ds^2 = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} dx^\mu dx^\nu \tag{2.4}$$

The summation sign is suppressed when one uses the *Einstein summation convention*. For example, in spherical polar coordinates we have $x = (t, r, \theta, \varphi)$ thus the metric reads

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{2.5}$$

or $g_{\mu\nu} = \text{diag}(-1, 1, r^2, r^2 \sin^2\theta)$ which is a flat metric[7] since it can be rewritten using flat coordinates.

[6]It is customary to use Greek letters to denote spacetime coordinates, whereas the Latin letters are used for space or other coordinates.

[7]We can also check whether the metric is flat by considering the Riemann tensor which is a curvature tensor whose all components become zero if and only if the metric is flat.

When we deal with the curved spacetime due to the presence of matter, the Riemann tensor plays a major role as seen in GR. One very important equation in this subject is the *Einstein field equation*, a tensorial equation which takes the form

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.6}$$

$G_{\mu\nu}$ is an *Einstein tensor* which is symmetric and vanishes when spacetime is flat. $T_{\mu\nu}$ is the so-called *energy-momentum tensor* which can be thought of as a source for the gravitational field. It is a divergenceless tensor due to the conservation of energy, namely $\nabla^{\mu} T_{\mu\nu} = 0$. The proportionality constant is 8π since we use the natural units, otherwise it would be $\frac{8\pi G}{c^4}$. Mathematically, the Einstein tensor is given by

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{2.7}$$

where $R_{\mu\nu}$ is called *Ricci tensor* which is a contraction of the Riemann tensor ($R_{\mu\nu} = R^{\gamma}_{\mu\gamma\nu}$), and R is a curvature scalar obtained from the Ricci tensor, hence called *Ricci scalar*. The full form of the Einstein equation has an extra term owing to the *Cosmological constant* (Λ) which has been found recently to be an extremely tiny number but non-zero. It reads

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.8}$$

The significance of the cosmological constant is involved mostly in the context of cosmology in which one studies the fate of the universe. The cosmological constant will appear again when we discuss the BTZ black hole in Section 2.2.

2.1 Black hole metrics

The geometry of a spherically symmetric vacuum, i.e. vacuum spacetime outside the spherical black hole is the *Schwarzschild geometry* describable in terms of the *Schwarzschild metric*[8]

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.9}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. The Schwarzschild metric is indeed a gravitational field and seems to have a singularity at the surface where $r = 2M$ due to its coordinates, in which space and time change their meanings. To get an idea of the magnitude of the Schwarzschild radius we note that for the Earth it is 0.1 cm and for the star of the size of the Sun, it is about 3 kilometers. The singularity at $r = 2M$ is however removable by chosing coordinates cleverly, an alternative to that is the *Kruskal-Szekeres coordinates* found in 1960, which represents the spacetime more properly. The surface of the black hole is entitled an "*event horizon*" for the fact that nothing can be seen beyond it. Only region on and outside the the black hole's surface, $r \geq 2M$ is observationally relevant. The Schwarzschild metric is necessarily

[8]This vacuum solution is always static in the exterior region of the black hole, c.f. Birkhoff's theorem.

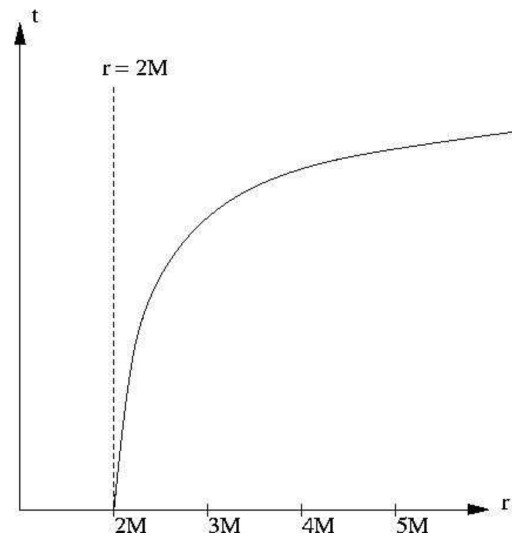

Figure 2.2: A trajectory of a particle or signal approaching the Schwarzschild black hole as represented in the Minkowski spacetime. The surface $r = 2M$ is the event horizon, nothing inside surface is able to escape. This trajectory is given by solving the Schwarzschild metric $ds^2 = 0$ (null geodesic) with $d\Omega^2 = 0$.

asymptotically flat[9], that is, for large r,

$$ds^2 \approx -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2 \tag{2.10}$$

and in addition to the Eq. (2.10) one can show that the Newtonian gravity is merely a limiting case of GR.

So far, we have seen that the Schwarzschild black hole depends only on the mass. This simple black hole is, astronomically, a collapse of an uncharged and non-rotating star with spherical symmetry. The gravitational collapse of a non-spherical star with non-zero net charge produces a somewhat different black hole which can be characterized by the mass M, intrinsic angular momentum or spin J and electric charge Q. It is found that the structure of a black hole is determined uniquely by just three parameters, i.e. M, J and Q once it is in the final state[10]. In this way it can be said that "black holes have no hair".[11] The black hole in the final state with only M and Q, has a gravitational field given by the *Reissner-Nordström metric* which takes the form

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2.11}$$

where M and Q are the total mass and charge respectively as measured by a distant observer. For the rotating uncharged black hole, (namely the black hole is characterized only by M and J) its geometry is given by the *Kerr metric* (usually represented

[9]This means that its metric approaches the flat metric far away from the black hole, namely $g_{ab} \to \eta_{ab}$ as $r \to \infty$.

[10]Final state here stands for stationary state in which everything is settled.

[11]Due to the fact that all traces of the matter that formed a black hole disappear except for M, J and Q thereby black holes look the same, as first introduced by J. Wheeler in 1960s.

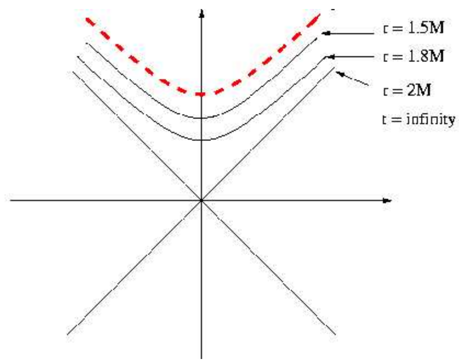

Figure 2.3: The Schwarzschild black hole in the Kruskal-Szekeres coordinates have no "coordinate singularity" hence represents the real spacetime where $r = 0$ is the singularity which is the broken thick line in the figure.

in the *Boyer-Lindquist coordinates*) using[12] $a = J/M$. The significance of the value of a plays a role when the extremal case is considered, i.e.

$$\frac{a}{M} = 0 \Rightarrow \text{There is no spin, hence reduced to the Schwarzchild case.}$$
$$\frac{a}{M} = 1 \Rightarrow \text{Extremal Kerr black hole is reached.}$$

The Kerr metric takes the form:

$$\begin{aligned} ds^2 &= -\frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2}dt\,d\varphi \\ &+\frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\,d\varphi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{2.12}$$

where

$$\Delta = r^2 - 2Mr + a^2 \tag{2.13}$$

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{2.14}$$

Its event horizons are (assuming that $a^2 < M^2$)

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{2.15}$$

The Kerr metric or Kerr solution is stationary and axially symmetric, and has double surfaces, i.e., outer and inner surfaces. In between the event horizon and the static limit lies the so-called *Ergosphere*[13] inside which nothing can remain stationary.

For the charged rotating black hole (named *Kerr-Newman black hole*), its geometry is given by the *Kerr-Newman metric* which is in the same form as Eq. (2.12) but with

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \tag{2.16}$$

The event horizons of the Kerr-Newman black hole are

$$r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2} \tag{2.17}$$

[12] J/M is refered to as angular momentum per unit mass.
[13] Ergo means energy.

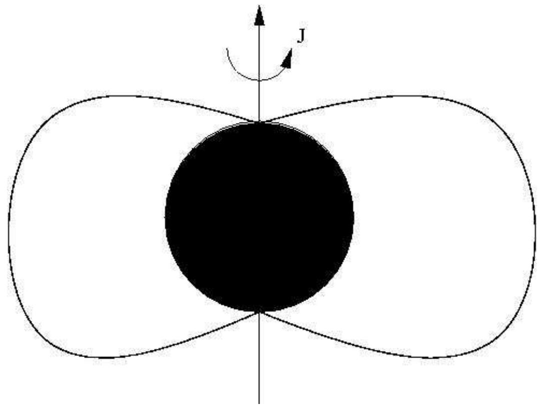

Figure 2.4: A rough sketch of the Kerr black hole which is surrounded by an ergosphere. The ergosphere is a region inside which nothing can remain stationary. The angular momentum of the Kerr black hole is denoted by J.

for $a^2 < M^2 + Q^2$. The extremal Kerr-Newman black hole is reached when $a^2 = M^2 + Q^2$.

It is notable that one would hardly observe charged black holes in nature because the black hole is already discharged when it is in stationary state. The discharging process takes place very rapidly. The time it takes to become stationary is called the *characteristic* time which is approximately $10^{-5}\frac{M}{M_\odot}$ second. Therefore, it is obvious that the black hole of 100000 $M_\odot$ only requires approximately 1 second to reach its stationary state.

2.2 The BTZ Black Hole: The (2+1)-dimensional black hole

Banãdos, Teitelboim and Zanelli discovered, in 1992, the black hole solution of Einstein's equation with a negative cosmological constant, in 2+1 dimensions and without couplings to matter [2, 3]. This discovery was rather surprising as there was no speculation that there would exist a black hole solution in 2+1 dimensions at that time. The BTZ black hole is known as a simple toy model for a number of studies including the string and supergravity theory. The cosmological constant, Λ is written as $-l^{-2}$ in the BTZ black hole's metric which reads:

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2(N^\varphi dt + d\varphi)^2 \tag{2.18}$$

where

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad N^\varphi = -\frac{J}{2r^2} \tag{2.19}$$

with $-\infty < t < \infty$, $0 < r < \infty$ and $0 \leq \varphi \leq 2\pi$. $N^2(r)$ and N^φ are the squared lapse and angular shift respectively.

The event horizons can be obtained from $N^2(r) = 0$ and takes the form:

$$r_\pm = l\sqrt{\frac{M}{2}(1 \pm \Delta)} \tag{2.20}$$

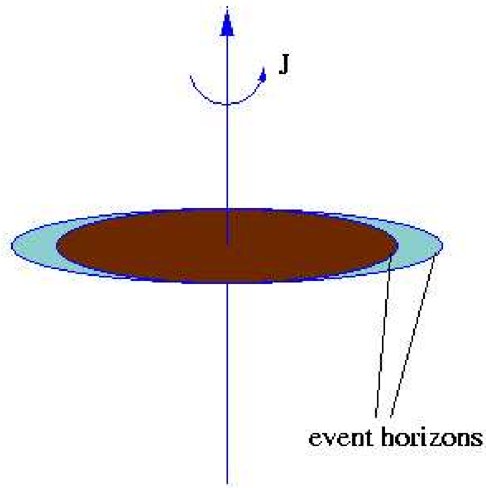

Figure 2.5: The 2+1 dimensional BTZ black hole. This black hole can be visualized as a circular disc with spin J. Its event horizon depends on its mass.

where

$$\Delta = \sqrt{1 - \left(\frac{J}{Ml}\right)^2} \tag{2.21}$$

with imposed conditions that

$$M > 0 \text{ and } |J| \leq Ml \tag{2.22}$$

In the extremal case $J = |Ml|$, the two event horizons coincide. Note that l is the radius of curvature which provides the length scale in order to have dimensionless mass. If one lets l grows very large the black hole exterior is pushed away to infinity and one will be left with the inside [2]. The BTZ black hole is similar to its 3+1 counterpart, the Kerr solution. The BTZ black hole has an ergosphere, namely $r_{\text{erg}} = l\sqrt{M}$ and an upper bound in angular momentum for any given mass. The spacetime geometry of the black hole is one of constant negative curvatures, so it is locally that of anti-de Sitter (adS) space. The BTZ black hole can only differ from the adS in its global properties [3]. The BTZ black hole is however different from its counterpart by the fact that it has a positive heat capacity instead of the negative heat capicity. Therefore its thermodynamic properties are well-defined and simple.

If we, for the sake of simplicity, set $\Lambda = -1$ (hence $l = 1$) for the spinless BTZ black hole, its metric is then reduced to

$$ds^2 = -(-M + r^2)dt^2 + \left(-M + r^2\right)^{-1} dr^2 + r^2 d\varphi^2 \tag{2.23}$$

The metric above is singular when $r = \sqrt{M}$, which is an event horizon of the spinless BTZ black hole.

2.3 Black holes and Thermodynamics

Black holes are found to be intimately related to ordinary thermodynamics in the sense that their mechanical laws are seemingly identical to the laws of the thermody-

Thermodynamic system	**Black hole**
Temperature, T	Surface gravity, κ
Energy, E	Black hole's mass, M
Entropy, S	Area of event horizon, A

Table 2.1: Analogy between thermodynamic parameters and black hole's parameters.

namics. In 1971 Stephen Hawking [14] stated that the area, A of the event horizon of a black hole can never decrease (but can remain constant) in any process:

$$\Delta A \geq 0 \tag{2.24}$$

The area of the event horizon increases when (1) mass increases and (2) spin decreases. It was later noted by Bekenstein [5] that this result is analogous to the statement of the ordinary second law of thermodynamics, namely that the total entropy, S of a closed system never decreases in any process:

$$\Delta S \geq 0 \tag{2.25}$$

With these arguments it is legitimate to establish the laws of *black hole mechanics* in parallel to the laws of ordinary thermodynamics by using parameters of the black hole (see Table 2.1) as follows:

`Zeroth law:` The event horizon is described by a quantity, κ, the surface gravity which is constant over the event horizon. The surface gravity is related to the physical temperature of the black hole[14] (*Hawking temperature*) by

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi} \tag{2.26}$$

For the special case of the Schwarzschild black hole, where $\kappa = 1/(4GM)$, the Hawking temperature becomes:

$$T_{\mathrm{H}} = \frac{\hbar}{8\pi G k_{\mathrm{B}} M} \approx 6.2 \times 10^{-8} \frac{M_{\odot}}{M}\ \mathrm{K} \tag{2.27}$$

So this is utterly negligible for solar-mass black holes – the black hole absorbs much more from the microwave background radiation than it radiates itself. In the case of the rotating "Kerr" black hole, the Hawking temperature is reduced by the rotation, explicitly

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi k_{\mathrm{B}}} = 2\left(1 + \frac{M}{\sqrt{M^2 - a^2}}\right)^{-1} \frac{\hbar}{8\pi M k_{\mathrm{B}}} < \frac{\hbar}{8\pi M k_{\mathrm{B}}} \tag{2.28}$$

where $a = J/M$. For the charged non-rotating "Reissner-Nordström" black hole, one has

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi k_{\mathrm{B}}} = \left(1 - \frac{Q^4}{r_+^4}\right) \frac{\hbar}{8\pi M k_{\mathrm{B}}} < \frac{\hbar}{8\pi M k_{\mathrm{B}}} \tag{2.29}$$

[14] $T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi k_B}$ when natural units are not used.

Thus, electric charge also reduces the Hawking temperature. As a conclusive remark one can safely say that the Hawking radiation plays no role in the case of large-sized black holes. The only type of black hole where one can hope to observe such the radiation is the so-called mini black hole, which is believed to have existed in the primordial stage of the Universe.

`First law:` This law deals with the mass (energy) change, dM when a black hole switches from one stationary state to another.

$$dM = \left(\frac{\kappa}{8\pi}\right) dA + \text{“work terms”} \tag{2.30}$$

or

$$dM = T_{\mathrm{H}}\, dS_{\mathrm{bh}} + \text{“work terms”} \tag{2.31}$$

It is readily seen that the above equations are analogous to the first law of thermodynamics, i.e.

$$dE = TdS + \text{“work terms”} \tag{2.32}$$

And the entropy of the black hole is thus represented by a quarter of the area of the event horizon, that is

$$S_{\mathrm{bh}} = \frac{A}{4} \tag{2.33}$$

The factor $\frac{1}{4}$ was indeed found by Hawking [16] based on the application of the quantum field theory to the black holes which shows that they will absorb and emit particles as if they were thermal bodies with the Hawking temperature, T_{H}. S_{bh} is sometimes called *Bekenstein-Hawking* entropy in order to honor their discoveries. The “work terms” are given differently depending on the type of the black holes. For the Kerr-Newman black hole family, the first law would be

$$dM = \left(\frac{\kappa}{8\pi}\right) dA + \Omega\, dJ + \Phi\, dQ \tag{2.34}$$

where Ω is the angular velocity of the hole and Φ is the electric potential which are defined by

$$\Omega = \frac{\partial M}{\partial J} \tag{2.35}$$

$$\Phi = \frac{\partial M}{\partial Q} \tag{2.36}$$

`Second law:` In any classical process, the area of the event horizon does not decrease

$$dA \geq 1 \tag{2.37}$$

nor does the black hole's entropy, S_{bh}. The second law of black hole mechanics can, however, be violated if the quantum effect is taken into account, namely that the area of the event horizon can be reduced via Hawking radiation. Note that the proof of this depends on the *Cosmic censorship conjecture*[15]. It is essential

[15] The cosmic censorship conjecture was made by the British mathematician Roger Penrose, which states that in the universe black holes enshroud singularities so that no information about singularities can reach an outside observers [30]. This issue was seriously investigated by Christodoulou and Hawking. And this is indeed the number-one open question in classical GR.

Law	**Thermodynamic system**	**Black hole**
Zeroth law	T constant on a body in thermal equilibrium	κ constant over a black hole's event horizon
First law	$dE = T\,dS - p\,dV + \mu\,dN$	$dM = \frac{\kappa}{8\pi}\,dA + \Omega\,dJ + \Phi\,dQ$
Second law	$dS \geq 0$	$dA \geq 0$
Third law	$T = 0$ cannot be reached	$\kappa = 0$ cannot be reached

Table 2.2: Analogy between the laws of thermodynamic and the laws of black-hole mechanics.

that the black hole radiation is thermal in nature, therefore generates a rise in entropy in the surrounding region. The *generalized entropy*, S' was introduced by Bekenstein [5, 6, 36] to account for this sort of entropy. It is defined as the sum of the black hole's entropy, S_{bh} and the entropy of the surrouding matter, S_{m}

$$S' = S_{\text{bh}} + S_{\text{m}} \tag{2.38}$$

This statement is known as the *Generalized Second Law* (GSL):

$$\Delta S' \geq 0 \tag{2.39}$$

The ordinary second law seems to fail when the matter is dropped into a black hole because according to classical GR, the matter will disappear into a spacetime singularity, in this manner the total entropy of the universe decreases as there is no compensation for the lost entropy. The virtue of the GSL keeps the law of entropy valid as the total entropy of the universe still increases when that matter falls into the black hole.

It is natural to question the magnitude of the entropy of the black hole. When expressed dimensionfully, the Bekenstein-Hawking entropy reads:

$$S_{\text{bh}} = \frac{k_{\text{B}} A}{4G\hbar} \tag{2.40}$$

for the Schwarzschild black hole, this yields [11]

$$S_{\text{bh}} = \frac{k_{\text{B}} \pi R_0^2}{G\hbar} \tag{2.41}$$

Numerically, the entropy of the sun is $S_{\odot} \approx 10^{57} k_{\text{B}}$ whereas a solar-mass black hole has an entropy of about $10^{77} k_{\text{B}}$ which is 20 orders of magnitude larger!

`Third law:` The limit $\kappa = 0$ cannot be reached within a finite time, in other words it is not possible how many processes we do, we will never reach reach the limit $\kappa = 0$. However, the extremal black holes, for example the Kerr black hole in which $a/M = 1$, do have $\kappa = 0$ thus zero temperature (absolute zero) but non-zero entropy. To actually reduce the surface gravity to zero is merely an idealized case because it is forbidden by the Cosmic censorship conjecture.

As far as physical theories are concerned, the laws of black hole mechanics have only been phenomenological thermodynamical laws [11], thereby a number of *open questions* can be raised, such as [17, 20, 37]:

- Is S_{bh} real or subjective?
- Where does it appear—on or near the horizon or deep in the hole?
- At what stage in the black hole's evolution is it created—immediately upon formation by gravitation collapse, or only gradually over the long course of evolution?
- What is the dynamical mechanism that makes S_{bh} a universal function, independent of the hole's past history or detailed internal condition?

  When the quantum effects are taken into account, one can ask:
- Can S_{bh} be derived from quantum mechanical considerations?
- Due to the effect of Hawking radiation (black hole evaporation), what happens to S_{bh} after the black hole has evaporated? Will all the information disappear after the evaporation?

These questions are somewhat embarrassing, because we do not know with our present knowledge how to answer them precisely. Nevertheless, it is hoped that success in modern theory of gravity, e.g., the quantum gravity or the string theory would be the key to answer—if not all—some of these open questions.

Chapter 3

Thermodynamic Fluctuation Theory

The idea that time may vary from place to place is a difficult one, but it is the idea Einstein used, and it is correct—believe it or not.

-R. Feynman

Thermodynamics (see [9] for example) is an old subject in physics and it is applicable to a wide variety class of systems. The ordinary laws of thermodynamics are not fundamental in their own right, but are laws that arise from the microscopic properties of the system. The utility of the thermodynamic laws is based on the fact that they have a universal validity, at least for most systems.

3.1 The role of entropy

Entropy is, in a sense, a measure of the disorder of a system. This quantity was first introduced by R. Clausius in 1850 as the amount of heat reversibly exchanged at a temperature T. Entropy undoubtedly plays a major role in thermodynamics and statistical mechanics. It is also the most characteristic extensive parameter (see Appendix A) in thermodynamics, namely when it is expressed in terms of other extensive parameters—it basically tells us the physics that underlines the system. Entropy is a part of the first and the second law of thermodynamics directly—it enters the first law to complete the differential representation of the internal energy, namely

$$dU = T\,dS - p\,dV + \mu\,dN \tag{3.1}$$

We can also expressed the entropy as a thermodynamic potential as

$$dS = \frac{1}{T}\,dU + \frac{p}{T}\,dV - \frac{\mu}{T}\,dN \tag{3.2}$$

which is just the differential form of the entropy. In this view, if the dependence of the entropy $S(U, V, N)$ on the variables U, V, N is known, then complete knowledge of all the thermodynamic parameters is obtained. Furthermore, the entropy tells us that for isolated systems (where $dQ_{\text{reversible}} = 0$) in equilibrium

$$dS = 0 \;\Leftrightarrow\; S = S_{\text{maximum}} \tag{3.3}$$

and for irreversible processes

$$dS > 0 \tag{3.4}$$

so in words it says that *the state of equilibrium is defined as the state of maximum entropy*. Now as an example we have the entropy of the ideal gas as [13]

$$S(N,T,p) = Nk_{\mathrm{B}}\left\{s_0(T_0,p_0) + \ln\left[\left(\frac{T}{T_0}\right)^{5/2}\left(\frac{p_0}{p}\right)\right]\right\} \tag{3.5}$$

where $s_0(T_0,p_0)$ is an arbitrary dimensionless function of the state (T_0,p_0), which is the result of the integration since Eq. (3.34) is derived from the second law of thermodynamics when we use $PV = Nk_{\mathrm{B}}T$ and $U = \frac{3}{2}Nk_{\mathrm{B}}T$. It is obvious that Eq. (3.34) gives us a full information about the ideal gas. The entropy of the ideal gas can also be expressed in terms of the other extensive parameters, i.e. $S(N,V,U)$ and it reads [7]

$$S(N,V,U) = Nk_{\mathrm{B}}\left\{s_0(N_0,V_0,U_0) + \ln\left[\left(\frac{N_0}{N}\right)^{5/2}\left(\frac{U}{U_0}\right)^{3/2}\left(\frac{V}{V_0}\right)\right]\right\} \tag{3.6}$$

From this equation, all equations of state of the ideal gas can be obtained by partial differentiation. Therefore by differentiating Eq. (3.6) with respect to the internal energy yields

$$\left(\frac{\partial S}{\partial U}\right)_{N,V} = \frac{1}{T} = \frac{3}{2}Nk\frac{1}{U} \Rightarrow U = \frac{3}{2}NkT \tag{3.7}$$

whereas differentiating it with respect to the volume we get the equation of state:

$$\left(\frac{\partial S}{\partial V}\right)_{N,U} = \frac{p}{T} = Nk\frac{1}{V} \Rightarrow pV = NkT \tag{3.8}$$

From statistical mechanics the entropy of the system is given by the natural logarithm of the number of microscopic states[1] Ω which reads (with the Boltzmann's constant being unity):

$$S = \ln\Omega \tag{3.9}$$

The microstate Ω is a function of the macrostate, i.e., $\Omega(U,V,N)$ hence the entropy is a function of these variables. Eq. (3.9) is very important for it provides the basic connection between macroscopic thermodynamics (entropy) and statistical microscopic physics (number of microscopic states). Notice that $S = 0$ when $\Omega = 1$, thus there is only one exact microstate, hence no disorder—and no entropy is created.

3.2 Fluctuations

Physical quantities which describe a macroscopic body in equilibrium are, almost always, close to their mean values. However there are certain small deviations from the mean values, which is the natural behavior of the system. These deviations are known as *thermodynamic fluctuations*. The problem that arises is to find the probability distribution of these deviations.

[1] A microstate is the specification of the quantum numbers of all the atomic constitutents of the subsystem. A microstate generally determines uniquely a macrostate, but not conversely.

When Eq. (3.9) is inverted[2] one gets

$$\Omega = e^S \tag{3.10}$$

which is the starting point of thermodynamic fluctuation theory indeed as it was first done by Einstein. We, however, preferably denote P as the probability distribution, i.e. $P \propto e^S$, we can Taylor expand the entropy about the fluctuation quantity x only up to the second order as

$$S(x) = S(0) - \frac{1}{2}\beta x^2 \tag{3.11}$$

where $\beta = -\left(\frac{\partial^2 S}{\partial x^2}\right)_0$ and $S(0) = 0$ since the entropy S has a maximum for $x = 0$ as $\frac{\partial S}{\partial x} = 0$. Substituting Eq. (3.11) into Eq. (3.10) yields:

$$P(x) = A\, e^{-\frac{1}{2}\beta x^2} \tag{3.12}$$

In differential form it reads:

$$P(x)\, dx = A\, e^{-\frac{1}{2}\beta x^2}\, dx \tag{3.13}$$

The constant A is given by the normalization condition that $\int P(x)\, dx = 1$. The integration limit is over all space, i.e., from $-\infty$ to ∞. This constant is found to be $\sqrt{\beta/2\pi}$ by Gaussian integration formula. Thus the probability distribution of the various values of the fluctuation reads:

$$P(x) = \sqrt{\frac{\beta}{2\pi}}\; e^{-\frac{1}{2}\beta x^2} \tag{3.14}$$

This probability distribution is catagolized as a *Gaussian distribution* . It reaches a maximum when $x = 0$ and decreases rapidly and symmetrically as $|x|$ increases. The mean squared fluctuation is defined as

$$\left\langle x^2 \right\rangle = \int dx\, P(x) x^2 = \frac{1}{\beta} \tag{3.15}$$

We can then write the Gaussian distribution as

$$P(x)dx = \frac{1}{\sqrt{2\pi \left\langle x^2 \right\rangle}} \exp\left(-\frac{x^2}{2\left\langle x^2 \right\rangle}\right) dx \tag{3.16}$$

It is then readily seen that, the smaller the $\left\langle x^2 \right\rangle$, the sharper the maximum of $P(x)$, which is the characteristics of the Gaussian distribution. Let us now consider the Gaussian distribution for more than one variable, namely we will determine simultaneous deviation of several thermodynamic quantities from their mean values. We define the entropy $S(x_1, \ldots, x_n)$ as a function of the quantities of a simultaneous deviation, and Taylor expand S in the same manner as done in Eq. (3.11) to the second-order, thus

$$S - S_0 = -\frac{1}{2} \sum_{i,j=1}^{n} \beta_{ij} x_i x_j \tag{3.17}$$

[2]This formula was first applied to the study of fluctuations by Einstein in 1907.

note that $\beta_{ij} = \beta_{ji}$. For simplicity we will omit the summation sign, we thus write

$$S - S_0 = -\frac{1}{2}\beta_{ij}x_i x_j \tag{3.18}$$

and the probability takes the form

$$P = A\exp\left(-\frac{1}{2}\beta_{ij}x_i x_j\right) \tag{3.19}$$

A is the normalization constant which is determined from the normalization condition namely,

$$\int dx_1 \int dx_2 \ldots \int dx_n\, P(x_1, \ldots, x_n) = 1 \tag{3.20}$$

After some algebraic manipulation (see [21], p. 335-338) we get

$$A = \frac{\sqrt{\beta}}{(2\pi)^{n/2}} \tag{3.21}$$

Therefore the desired form of the Gaussian distribution for more than one variable reads

$$P = \frac{\sqrt{\beta}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}\beta_{ij}x_i x_j\right) \tag{3.22}$$

where

$$\beta_{ij} = -\frac{\partial^2 S}{\partial x_i \partial x_j}, \quad \beta = |\beta_{ij}| \tag{3.23}$$

3.3 Thermodynamics and Geometry

Ruppeiner [28] in 1979 proposed a new way to study thermodynamics by using a Riemannian geometrical model, in other words, he claims that thermodynamic systems can be represented by Riemannian geometry and certain statistical properties can be derived from the model. This geometrical model is based on the inclusion of the theory of fluctuations into the axioms of equilibrium thermodyanmics, namely there exist equilibrium states which can be represented by points in two-dimensional surface (manifold) and the distance between these equilibrium states is related to the fluctuation between them. This concept is associated to probabilities, i.e., the less probable a fluctuation between states, the further apart they are. This can be recognized if one considers β_{ij} in the distance formula (line element) between the two equilibrium states

$$ds^2 = \sum_{ij} \beta_{ij} dx^i dx^j = \beta_{ij} dx^i dx^j \tag{3.24}$$

where the matrix of coefficients β_{ij} is the metric tensor, which is symmetric. We call a manifold with a rule for distance in the form of Eq. (3.24) a Reimannian manifold. If we now define

$$X_i = \frac{\partial S}{\partial x^i} = \beta_{ij} x_j \tag{3.25}$$

we will find that [21]

$$\langle x_i x_j \rangle = \beta^{ij} = \langle X^i X^j \rangle \tag{3.26}$$

which is the second moment for the fluctuations or the pair correlation function. We refer the variables X^i as *thermodynamically conjugate* to x_i and we will assign S_{ij} to be the metric tensor intead of β_{ij}. The probability distribution can then be rewritten as

$$P(X) = \frac{\sqrt{S}}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2} S_{ij} X^i X^j\right) \tag{3.27}$$

where

$$S_{ij} = -\frac{\partial^2 S}{\partial X^i \partial X^j}, \quad S = |S_{ij}| \tag{3.28}$$

It is not entirely clear from the outset how this can be used to describe thermodynamic systems, but if we limit ourselves to the coordinates which are extensive parameters in ordinary thermodynamics, we will be able to obtain some thermodynamic properties according to Ruppeiner [28]. The following sections will discuss this:

3.3.1 Ruppeiner metrics

The Ruppeiner metric is defined as the second derivatives of the entropy of the system with respect to the internal energy and other extensive charges (thermodynamic parameters) , namely $Q^i = (M, N^a)$, it reads:

$$S_{ij} = -\frac{\partial^2 S}{\partial Q^i \, \partial Q^j} \tag{3.29}$$

N^a is a conserved quantity and $a = 1, \ldots, n$. An interesting fact about this metric is that its inverse form is

$$S^{ij} = \frac{\partial^2 \Gamma}{\partial \beta_i \beta_j} \tag{3.30}$$

where $\beta_i = \left(\frac{1}{T}, -\frac{\mu_a}{T}\right)$ which are the conjugate variables. Γ is the Legendre transformation of the entropy S, i.e.

$$\Gamma = \frac{G}{T} = -S + \frac{M}{T} - N^a \frac{\mu_a}{T} \tag{3.31}$$

where G is the Gibbs free energy defined as:

$$G = M - TS - \mu_a N^a \tag{3.32}$$

The sum of the entropy and its Legendre transformation is equal to the product of the extensive charges and the conjugate variables, namely

$$S + \Gamma = \beta_i Q^i \tag{3.33}$$

The Ruppeiner metric is physically meaningful in the equilibrium thermodynamic fluctuation theory, i.e., the *thermodynamic Riemannian curvature*, in short, *thermodynamic curvature* for any given thermodynamic state tells us about the underlying statistical structure. Calculations of the Ruppeiner metrics as well as their Riemannian curvatures for different thermodynamic systems have been carried out. As an example, the ideal gas at fixed volume has the Ruppeiner metric in the following form [29]:

$$ds^2 = \left(\frac{C_V}{T^2}\right) dT^2 + \left(\frac{1}{T} \frac{V}{\rho^2 K_T}\right) d\rho^2 \tag{3.34}$$

where C_V is the heat capacity at constant volume and K_T is the isothermal compressibility of the system. Notice that we are in the coordinates (T, ρ) where $\rho(\equiv N/V)$, is the density. To figure out whether this system is interacting or not, we can do so either by coordinate transformation (i.e., rewriting it in the form of Euclidean metric, which assures the flat space, i.e. the zero curvature) or computing the scalar curvature of the metric in Eq. (3.34). It is not always possible to do the former method if the given metric is very complicated, the formula for the latter is given in Appendix B (Eq. B.9). It turns out—for the ideal gas—that it is a non-interacting system, which we of course know from our exisiting knowledge. The curvature scalar for the ideal gas with the particle number rather than the volume as the fixed scale is also zero as found by Nulton and Salamon [26].

3.3.2 Weinhold metrics

Weinhold [38] was the first to introduce the geometrical concept into thermodynamics. In his approach the Riemannian metric is considered in the energy representation where extensive parameters of the subsystem is given by

$$N^i = (S, N^a) \tag{3.35}$$

The Weinhold metric W_{ij} is defined as the second derivatives of the internal energy (in this case, mass M) with respect to the entropy and N^i as given in Eq. (3.35), explicitly

$$W_{ij} = \frac{\partial^2 M}{\partial N^i \partial N^j} \tag{3.36}$$

The contravariant form of the Weinhold metric is given by

$$W^{ij} = -\frac{\partial^2 G}{\partial \mu_i \partial \mu_j} \tag{3.37}$$

Where

$$G = M - TS - \mu_a N^a \tag{3.38}$$

with $\mu_i = (T, \mu_a)$ and $a_{=}1, \ldots, n$. Note that, in general, for black holes these metrics will not be positive definite due to the fact that the ordinary black holes have negative heat capacities [12]. The negative heat capacity is a property of the isolated self-gravitating systems such as stars and astronomical systems [22]. Stars and black holes display the same phenomenon in this aspect, their temperatures increase as they lose energy, which is in accordance with the virial theorem.

Remarkably, the Weinhold and the Ruppeiner metrics are conformally related via temperature T as the conformal factor, mathematically

$$W_{ij}\, dN^i dN^j = T\, S_{ij}\, dQ^i dQ^j \tag{3.39}$$

The conformal relation will be of great importance, when the Ruppeiner metric seems to be very difficult to handle, by this we mean that instead of computing the Ruppeiner metric, we will work out the Weinhold metric and turn it into the Ruppeiner metric via T, which is defined as $T = \frac{\partial M}{\partial S}$. It is worth noting that the Ruppeiner metric obtained by using the conformal relation will be in the Weinhold coordinates, but its geometry is the Ruppeiner geometry.

Chapter 4

Results

The opposite of a correct statement is a false statement. But the opposite of a profound truth may well be another profound truth. –N. Bohr

Having introduced the Ruppeiner and the Weinhold metrics in Chapter 3 we can now compute them for 4 different families of black holes. In order to do so, we need an expression for the entropy (for the Ruppeiner metrics) and the mass (for the Weinhold metrics). We will also use the mass formula to calculate the intensive parameters such as temeparature T, angular velocity Ω and the electric charge Φ as well as the heat capacity C for each black hole. Some thermodynamic curvature scalars obtained from the Ruppeiner metrics will be plotted and discussed.

Smarr [32] in 1973 found the *mass formula* which contains all the information about the thermodynamic state of the Kerr-Newman black hole as

$$M = \sqrt{\frac{1}{4}\left(\frac{A}{4\pi}\right) + \frac{4\pi}{A}\left(J^2 + \frac{Q^4}{4}\right) + \frac{Q^2}{2}} \tag{4.1}$$

where J and Q are the black hole's spin and the electric charge respectively. A is the surface (event horizon) area of the black hole which is expressible in terms of the hole's entropy as

$$S = \frac{1}{4}k_{\mathrm{B}}A \tag{4.2}$$

by inserting $A = \frac{4S}{k_{\mathrm{B}}}$ in Eq. (4.1) with $k_{\mathrm{B}} = 1/\pi$; it is found that the algebra becomes nicely simplified—this simplicity turns out to be extremely useful when we proceed to compute the Ruppeiner and the Weinhold metrics of these black holes as it reduces the degree of complication and tediousness. The simplified Smarr's mass formula takes the form:

$$M = \sqrt{\frac{S}{4} + \frac{1}{S}\left(J^2 + \frac{Q^2}{4}\right) + \frac{Q^2}{2}} \tag{4.3}$$

We will use this relation for obtaining certain thermodynamic properties of the Reissner-Nordström, the Kerr and the Kerr-Newman black holes. We will start with a theoretical BTZ black hole.

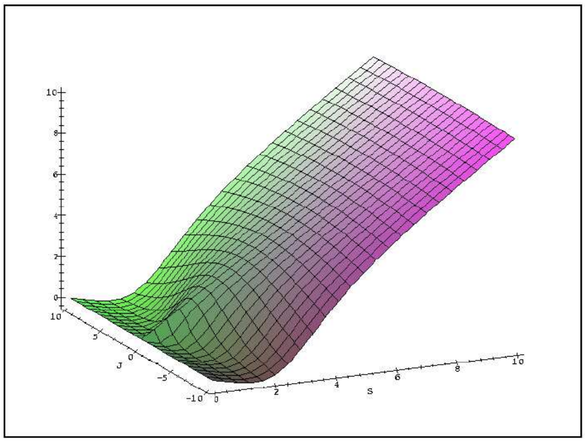

Figure 4.1: Heat capacity of the BTZ black hole as a function of S and J. The $C(S, J)$ is a truly positive function for any values of S and J. It is notable that at the extremal limit $J = 2S^2$, which is the same as $J = M$ the temperature vanishes—hence there is no heat capacity.

4.1 BTZ black hole

For the sake of simplicity we will use $l = 1$ (that is we use $l = 1$ in Eq. (2.20)). Therefore we have the event horizons of the BTZ black hole in the form:

$$r_{\pm} = \sqrt{\frac{M}{2}(1 \pm \Delta)} \tag{4.4}$$

where

$$\Delta = \sqrt{1 - \left(\frac{J}{M}\right)^2} \tag{4.5}$$

The entropy of the BTZ black hole (note that we use $k_{\mathrm{B}} = 2/\pi$ for this particular case) is

$$S = \frac{A}{2\pi} = \frac{L}{2\pi} = \frac{2\pi r_+}{2\pi} = \sqrt{\frac{M}{2}(1 + \Delta)} \tag{4.6}$$

where L is the length of the horizon r_+. By solving Eq. (4.6) for M we have the mass in terms of the entropy for the BTZ black hole as

$$M = S^2 + \frac{J^2}{4S^2} \tag{4.7}$$

The temperature of the BTZ black hole is given by

$$T = \frac{\partial M}{\partial S} = 2S - \frac{J^2}{2S^3} \tag{4.8}$$

and its angular velocity is

$$\Omega = \frac{\partial M}{\partial J} = \frac{J}{2S^2} \tag{4.9}$$

Its heat capacity takes the form:

$$\begin{aligned} C &= \frac{\partial M}{\partial T} = T\frac{\partial S}{\partial T} = \frac{T}{\left(\frac{\partial T}{\partial S}\right)} = \frac{T}{\left(\frac{\partial^2 M}{\partial S^2}\right)} \\ \Rightarrow C &= \frac{S(4S^4 - J^2)}{4S^4 + 3J^2} \end{aligned} \tag{4.10}$$

It is readily seen that the heat capactity of the BTZ hole is positive. We have found that for this black hole it is more convenient to evaluate the Weinhold metric and use the conformal relation to obtain the Ruppeiner metric. The Weinhold line element reads

$$ds_W^2 = \left(2 + \frac{3J^2}{2S^4}\right) dS^2 - \left(\frac{2J}{S^3}\right) dSdJ + \left(\frac{1}{2S^2}\right) dJ^2 \tag{4.11}$$

The Weinhold metric turns out to be a non-flat metric but we are not interested in its curvature scalar, so we proceed to calculate the Ruppeiner metric for the BTZ black hole by using $ds_R^2 = \frac{1}{T} ds_W^2$ which takes the form:

$$ds_R^2 = \left(\frac{4S^4 + 3J^2}{(4S^4 - J^2)S}\right) dS^2 - \left(\frac{2J}{4S^4 - J^2}\right) dSdJ + \left(\frac{S}{4S^4 - J^2}\right) dJ^2 \tag{4.12}$$

and it is diagonalizable as

$$ds_R^2 = \left(\frac{1}{S}\right) dS^2 + \left(\frac{S}{1 - u^2}\right) du^2 \tag{4.13}$$

with

$$u = \frac{J}{2S^2}, \quad -1 \leq u \leq 1 \tag{4.14}$$

This metric yields a flat Riemannian curvature, i.e., the space of thermodynamic state is a flat space.

4.2 Reissner-Nordström black hole

From the mass formula in Eq. (4.3), the Reissner-Nordström black hole has

$$M = \sqrt{\frac{S}{4} + \frac{1}{S}\frac{Q^2}{4} + \frac{Q^2}{2}} \tag{4.15}$$

fortunately it can be simplified as

$$M = \frac{\sqrt{S}}{2}\left(1 + \frac{Q^2}{S}\right) \tag{4.16}$$

The temperature and the heat capacity for the Reissner-Nordström black hole are as follow:

$$T = \frac{\partial M}{\partial S} = \frac{1}{4\sqrt{S}}\left(1 - \frac{Q^2}{S}\right) \tag{4.17}$$

and

$$C = -2S\frac{\left(1 - \frac{Q^2}{S}\right)}{\left(1 - \frac{3Q^2}{S}\right)} \tag{4.18}$$

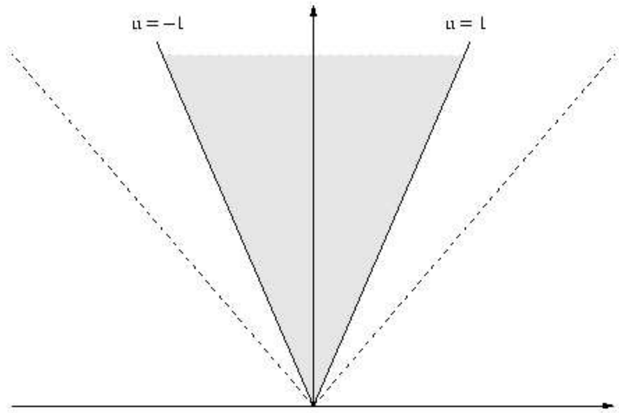

Figure 4.2: The Ruppeiner metric for the RN black hole as seen in Minskowski space when Rindler coordinates are used. The shaded area represents the Lorentzian metric, i.e., the metric of $(-+)$ signature.

and the electric potential takes a simple form:

$$\Phi = \frac{\partial M}{\partial Q} = \frac{Q}{\sqrt{S}} \tag{4.19}$$

It is obvious that the heat capacity for the Reissner-Nordström hole is a negative quantity. Notice that there is an absolute-zero temperature associated with the hole when $Q = \sqrt{S}$, which is the extremal limit. We next calculate the Weinhold metric and obtain

$$ds^2_W = \frac{1}{8S^{\frac{3}{2}}}\left[-\left(1 - \frac{3Q^2}{S}\right)dS^2 - 8QdSdQ + 8SdQ^2\right] \tag{4.20}$$

We can diagonalize this metric by using

$$\xi = \frac{Q}{\sqrt{S}}, \quad -1 \leq \xi \leq 1 \tag{4.21}$$

which yields

$$ds^2_W = \frac{1}{8S^{\frac{3}{2}}}\left[-(1-\xi^2)dS^2 + 8S^2 d\xi^2\right] \tag{4.22}$$

We can turn the Weinhold metric above into the the Ruppeiner metric in the same coordinates via the conformal relation as

$$ds^2_R = \left(\frac{-1}{2S}\right)dS^2 + \left(\frac{4S}{1-\xi^2}\right)d\xi^2 \tag{4.23}$$

which has the same properties as the Ruppeiner metric obtained directly from the entropy representation where

$$S(M,Q) = 2M^2 - Q^2 + 2M^2\sqrt{1 - \frac{Q^2}{M^2}} \tag{4.24}$$

$S(M,Q)$ is just an inversion of the mass formula in Eq. (4.16). By computing its curvature scalar, it is found that the Ruppeiner metric for the RN black hole is a

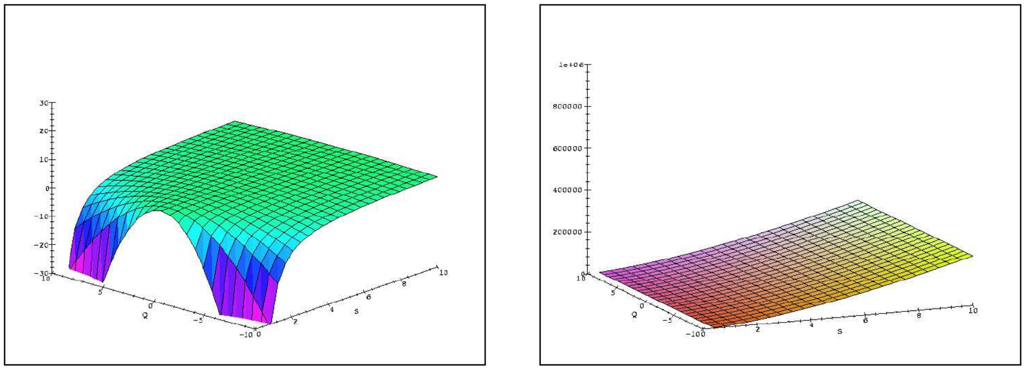

Fig.(A) Fig.(B)

Figure 4.3: Heat capacity of the RNadS black hole as a function of entropy S and electric charge Q for 2 different values of l, i.e., we have set $l = 1$ in Fig.(A) and $l = 0.01$ in Fig.(B). The heat capacity of the RNadS black hole vanishes at the extremal limit, whereas in the case of a small value of l (large Λ) as in Fig. (B) the heat capacity becomes linear in the entropy.

flat metric, which is the same as that of the BTZ black hole. We now introduce new coordinates

$$\tau = \sqrt{2S} \quad \text{and} \quad \sin\frac{\sigma}{\sqrt{2}} = \xi \tag{4.25}$$

The line element in Eq. (4.23) then reads

$$ds^2 = -d\tau^2 + \tau^2\, d\sigma^2 \tag{4.26}$$

This is a timelike wedge in a Minkowski space as seen in Rindler coordinates, see Fig. (4.2).

For the sake of completeness we consider also the Reissner-Nordström black hole with a negative cosmological constant (the Reissner-Nordström-anti-de Sitter or RNadS for short) and we use the mass formula [8]:

$$M^2 = \frac{A}{16\pi} + \frac{\pi}{4}(4J^2 + Q^4) + \frac{Q^2}{2} + \frac{J^2}{l^2} + \frac{A}{8\pi l^2}\left(Q^2 + \frac{A}{4\pi} + \frac{A^2}{32\pi^2 l^2}\right) \tag{4.27}$$

with $J = 0$ and $k_{\mathrm{B}} = 1/\pi$ (hence $S = \frac{A}{4\pi}$) it simply becomes

$$M = \frac{\sqrt{S}}{2}\left(1 + \frac{S}{l^2} + \frac{Q^2}{S}\right) \tag{4.28}$$

The temperature associated with the RNadS black hole is given by

$$T = \frac{1}{4\sqrt{S}}\left(1 + \frac{3S}{l^2} - \frac{Q^2}{S}\right) \tag{4.29}$$

which vanishes in the extremal limit, namely at

$$\frac{Q^2}{S} = 1 + \frac{3S}{l^2} \tag{4.30}$$

as well as the heat capacity for the RNadS black hole, which takes the form

$$C = 2S\left[\frac{S\left(\frac{3S}{l^2}+1\right)-Q^2}{S\left(\frac{3S}{l^2}-1\right)+3Q^2}\right] \tag{4.31}$$

The heat capacity of the RNadS black hole becomes linear in S in the limit of $l \to \infty$. We display heat capacities of this black hole for 2 different values of l in Fig. (4.3) and its electric charge is given by

$$\Phi = \frac{\partial M}{\partial Q} = \frac{Q}{\sqrt{S}} \tag{4.32}$$

We are now in the position to compute the Ruppeiner metric for the RNadS black hole; we will do so via the conformal relation as it is obviously simpler, in this case, to calculate the Weinhold metric directly from the mass formula in Eq. (4.28), we thus have

$$ds_W^2 = \frac{1}{8S^{\frac{3}{2}}}\left[-\left(1-\frac{3S}{l^2}-\frac{3Q^2}{S}\right)dS^2 - 8QdSdQ + 8SdQ^2\right] \tag{4.33}$$

this metric can be diagonalized using the same coordinates as in Eq. (4.25) which yields the diagonal metric in the form

$$ds^2 = \frac{1}{\left(1+\frac{3\tau^2}{2l^2}-\xi^2\right)}\left[-\left(1-\frac{3\tau^2}{2l^2}-\xi^2\right)d\tau^2 + 2\tau^2 d\xi^2\right] \tag{4.34}$$

This metric has a non-trivial geometry, namely there is a change of the signature of the metric at

$$\xi^2 = \frac{Q^2}{S} = 1 - \frac{3S}{l^2} \tag{4.35}$$

This corresponds to the stability properties of the thermodynamic system which change for sufficiently large black holes [1]. We show this in the Fig. (4.4). From the Ruppeiner metric we obtain the Riemannian curvature scalar in the form

$$R = \frac{9}{l^2}\frac{\left(\frac{3S}{l^2}+\frac{Q^2}{S}\right)\left(1-\frac{S}{l^2}-\frac{Q^2}{S}\right)}{\left(1-\frac{3S}{l^2}-\frac{Q^2}{S}\right)^2\left(1+\frac{3S}{l^2}-\frac{Q^2}{S}\right)} \tag{4.36}$$

which has a divergence in the extremal limit and along the curve where the metric changes its signature. The 3D plot of this is in Fig. (4.5). This RNadS black hole comes the ordinary RN black hole in the limit of $l \to \infty$.

4.3 Kerr black hole

The mass formula for the Kerr black hole is obtained by setting $Q = 0$ in Eq. (4.3) which reads

$$M = \sqrt{\frac{S}{4}+\frac{J^2}{S}} \tag{4.37}$$

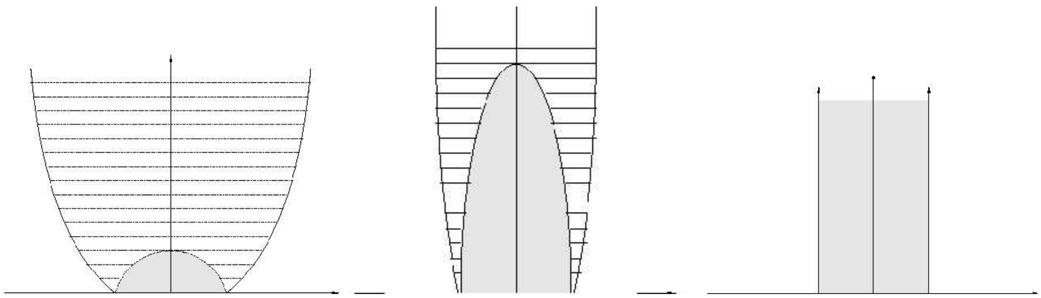

Fig.(A) Fig.(B) Fig.(C)

Figure 4.4: The RNadS black hole's Ruppeiner geometry—the shaded areas represent the Lorentian metrics while the strips are the Riemannian metric which is a positive definite metric with the $(++)$ signature. Fig.(A) is when $l = 1$ which is the when the cosmological constant $\Lambda = -1$, Fig.(B) represents the geometry when the value of l is large. In Fig.(C) we have $l \to \infty$ which means that the cosmological constant is vanishing thus it reduces to the case of RN black hole where its Ruppeiner metric is a Lorentzian metric.

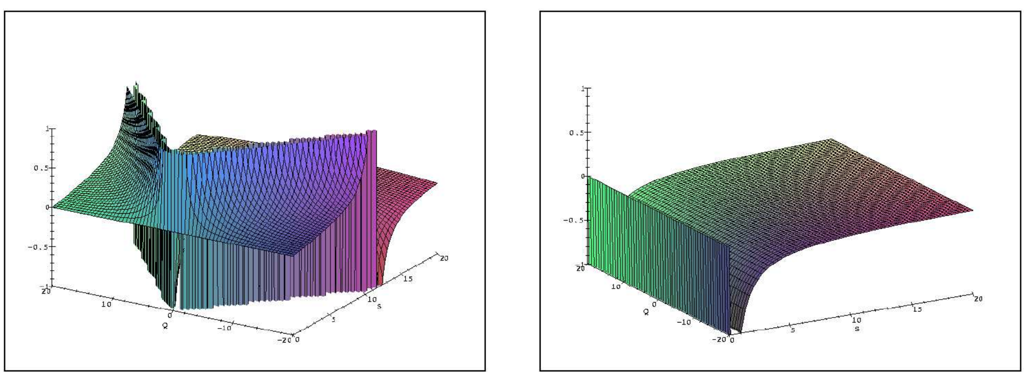

Fig.(A) Fig.(B)

Figure 4.5: The Riemannian curvatures for the RNadS black hole as a function of Q, S with 2 different values of l. In Fig.(A) we set $l = 1$ whereas Fig.(B) represents the curvature where $l = 0.001$. There are discontinuities in both cases along the extremal limits.

The temperature temperature takes the form:

$$T = \frac{\partial M}{\partial S} = \frac{1 - \frac{4J^2}{S^2}}{4\sqrt{S + \frac{4J^2}{S}}} \tag{4.38}$$

and it vanishes in the extremal limit, namely $S = 2J$ or in terms of J and M it is $J = M^2$. The angular velocity of the hole is given by

$$\Omega = \frac{\partial M}{\partial J} = \frac{2J}{S\sqrt{S + \frac{4J^2}{S}}} \tag{4.39}$$

On inversion of Eq. (4.37) we obtain the entropy in the form

$$S = 2M^2 + 2M^2\sqrt{1 - \frac{J^2}{M^4}} \tag{4.40}$$

hence we obtain the Ruppeiner metric in the following form

$$ds_R^2 = \frac{2}{\left(1 - \frac{J^2}{M^4}\right)^{\frac{3}{2}}} \left\{ -2\left[\left(1 - \frac{J^2}{M^4}\right)^{\frac{3}{2}} + 1 - \frac{3J^2}{M^4}\right] dM^2 - \left(\frac{4J}{M^3}\right) dM dJ + \left(\frac{1}{M^2}\right) dJ^2 \right\} \tag{4.41}$$

by the virtue of the coordinate transformation it is reduced to the diagonal form as

$$ds_R^2 = -2\left(1 + \frac{2}{\sqrt{1 - \eta^2}}\right) dM^2 + \left(\frac{2M^2}{(1 - \eta^2)^{\frac{3}{2}}}\right) d\eta^2 \tag{4.42}$$

where

$$\eta = \frac{J}{M^2} \tag{4.43}$$

It turns out that the Ruppeiner geometry of this black hole is curved—it scalar curvature is given by

$$R = \frac{1}{4M^2} \frac{\sqrt{1 - \frac{J^2}{M^4}} - 2}{\sqrt{1 - \frac{J^2}{M^4}}} \tag{4.44}$$

The above curvature scalar has a divergence in the extremal limit of the Kerr black hole, namely at $J = M^2$ which is where the extremal limit of the black hole.

4.4 Kerr-Newman black hole

The Kerr-Newman black hole has the mass formula given by Eq. (4.3), namely

$$M(S, J, Q) = \sqrt{\frac{S}{4} + \frac{1}{S}\left(J^2 + \frac{Q^2}{4}\right) + \frac{Q^2}{2}} \tag{4.45}$$

Thermodynamics of this black hole can be described by the variation in mass M as

$$dM = T\, dS + \Omega\, dJ + \Phi\, dQ \tag{4.46}$$

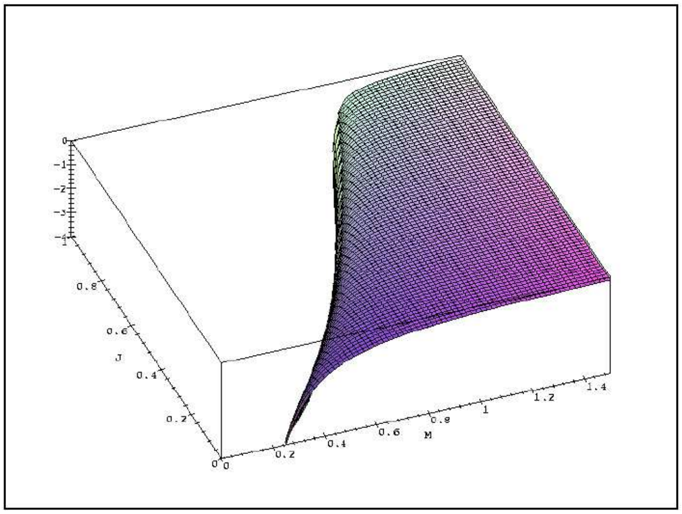

Figure 4.6: The curvature scalar for the Kerr black hole as a function of M and J. It is noticeable that a divergence of the curvature occurs along $J = M^2$ which is an extremal limit of this black hole .

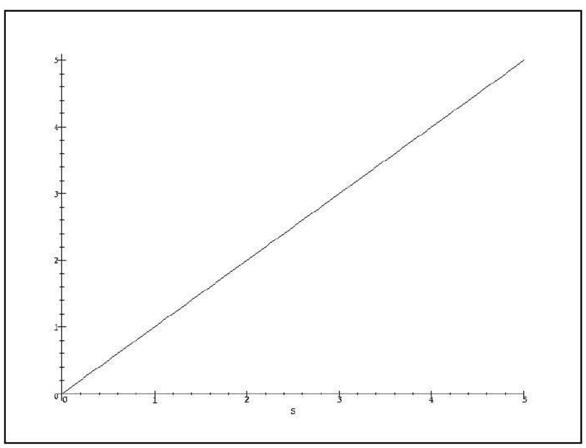

Figure 4.7: Heat capacity for the KN black hole for constant J and Q, we here set $J = Q = 1$.

which is indeed the second law of thermodynamics where the intensive parameters T, Ω and Φ are given by

$$T = -\frac{1}{4}\frac{-S^2 + 4J^2 + Q^4}{S\sqrt{S(4J^2 + S^2 + 2SQ^2 + Q^4)}} = \frac{1}{4}\frac{(S^2 - 4J^2 + Q^4)}{2MS^2} \tag{4.47}$$

with the entropy S being just the inversion of Eq. (4.46) which takes the form:

$$S = 2M^2 - Q^2 + 2M^2\sqrt{1 - \frac{Q^2}{M^2} - \frac{J^2}{M^4}} \tag{4.48}$$

The other intensive parameters are its angular velocity Ω

$$\Omega = \frac{2J}{\sqrt{S(4J^2 + S^2 + 2SQ^2 + Q^4)}} = \frac{J}{MS} \tag{4.49}$$

It is readily seen that the larger the mass and entropy of the black hole, the smaller the angular velocity, and as expected the angular velocity is proportional to the angular momentum of the black hole. The electric potential takes the following form:

$$\Phi = \frac{Q(S + Q^2)}{\sqrt{S(4J^2 + S^2 + 2SQ^2 + Q^4)}} = \frac{Q(S + Q^2)}{2MS} \tag{4.50}$$

This potential is reduced by mass and entropy of the black hole. As a matter of fact, the KN black hole has a fairly complicated form of heat capacity, therefore we will only show it graphically. The heat capacity is a function of S, J and Q—for simplicity we will consider it for constant angular momentum and charge. Thus by setting $J = Q = 1$ we have found that the heat capacity becomes linear in entropy, see plot in Fig. (4.7).

The Ruppeiner metric of the Kerr-Newman black hole is very complicated and too tedious to work out by hand, therefore we utilize a computer program (GRtensor) to ease the calculation. We will not present it here since it does not give any significance to the rest of this work. However we have found that the KN black hole has curved geometries of both the Ruppeiner and the Weindhold metrics—the Ruppeiner is found to be non-conformally flat [1].

Chapter 5

Conclusions and Speculations

Somewhere, there is something incredible waiting to be known.

–C. Sagan

In this thesis work, thermodynamic Riemannian geometries of various black hole families are studied, namely the Ruppeiner and Weinhold geometries. The Ruppeiner geometry is constructed on the basis of the fluctuation theory in equilibrium thermodynamics—and it is conformally related to the Weinhold geometry via temperature in the mathematical expression. It is, however, only the Ruppeiner geometry that tells us the statistical interaction underlying the system under consideration. Although the black hole is believed to be a thermodynamic system, its statistical mechanical structure is lacking[1], partly due to own limited knowledge of gravity at the quantum level. The Ruppeiner geometry for the BTZ and RN families is flat (zero curvature), as expected because of the simpler structure they possess The thermodynamic scalar curvature diverges in the extremal limit in the Kerr, RNadS and KN families that possess a curved geometry. Most interestingly perhaps, is the RNadS black hole family whose curvature is singular along the line where the stability properties change, and it coincides with the RN family in the limit of vanishing cosmological constant. The Weinhold geometry gives rise to no physical meaning—it is curved for the BTZ, RN and RNadS families whereas the Kerr family has a flat Weinhold geometry. For the sake of tidiness, the obtained results are tabulated in the following page. We believe that these results are sensible and simpler than what one might have expected, although no statistical physics of black holes can be derived from these findings at this point in time[2]. Finally, it is hoped that these results will find an interesting pattern in the future when the quantum theory of black holes is more concrete.

[1]There have been some results from string theory concerning the counting of the black hole microstates, for example, see [18, 33].

[2]As an encouraging remark, one may try to apply this calculation to black holes in string theory.

Black hole family	Ruppeiner metric	Weinhold metric
BTZ	Flat	Curved
Reissner-Nordström	Flat	Curved
Reissner-Nordström-anti-de-Sitter	Curved	Curved
Kerr	Curved	Flat
Kerr-Newman	Curved	Curved

Table 5.1: Geometries of the Ruppeiner and Weinhold metrics for various black holes.

Appendix A: Thermodynamics in a nutshell

The succint description of thermodynamics as given by W. Greiner [13] is:

> The task of thermodynamics is to define appropriate physical quantities (the *state quantities*), which characterize macroscopic properties of matter, the so-called *macrostate*, in a way which is as unambiguous as possible, and to relate these quantities by means of universally valid equations (the *equations of state* and the *laws of thermodynamics*).

We will give here some terminologies used in thermodynamics, important equations and descriptions of certain thermodynamic systems [9, 13, 40]:

- *Temperature*: In thermodynamics the temperature (or thermodynamic temperature) is defined as a measure of the average kinetic energy of the particles in a system. Adding heat to a system causes its temperature to rise. While there is no maximum theoretically reachable temperature, there is a minimum temperature, known as absolute zero, at which all molecular motion stops.

- *Extensive parameters* or *additive state quantities* are the quantities that are proportional to the amount of matter in a system, e.g. to the particle number or mass. Obvious examples are the volume and the energy.

- *Intensive parameters* or *intensive state quantities* are the quantities that are independent of the amount of matter and are *not* additive. Examples are: refractive index, density, temperature, pressure, chemical potential, etc. These quantities can be locally defined, i.e. they may vary spatially.

- *Equations of state*: We call certain relationships expressing intensive parameters in terms of the independent extensive parameters the "equations of state". For example

$$T = T(S, V, N_1, \ldots, N_r) \tag{A.1}$$

$$P = P(S, V, N_1, \ldots, N_r) \tag{A.2}$$

$$\mu = \mu(S, V, N_1, \ldots, N_r) \tag{A.3}$$

- *Heat capacities*: The heat capacity C of a substance is the amount of heat required to change its temperature by one degree, and it has a unit of energy per degree. The heat capacity is an extensive variable since a large quantity of matter will have a proportionally large heat capacity.

  The heat capacity at constant pressure is then defined by

$$C_P = \left(\frac{dQ}{dT}\right)_P = T\left(\frac{\partial S}{\partial T}\right)_P \tag{A.4}$$

  while at the constant volume it reads:

$$C_V = \left(\frac{dQ}{dT}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = T\left(\frac{\partial E}{\partial T}\right)_V \tag{A.5}$$

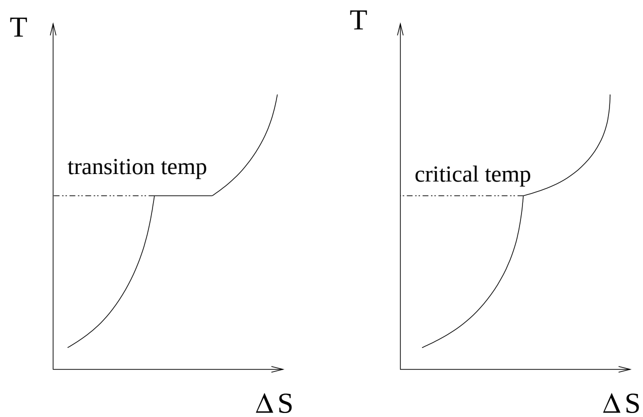

Figure A.1: Phase transitions in $T-S$ diagram. On the left is the phase transition of first order; it is obvious that there is a jump in entropy hence the divergence of the heat capacity. The phase transition of second order is shown on the right in the figure, there is a finite jump in entropy in this case. We assume the constant pressure for both cases.

- *Reversible processes*: These are processes which proceed over equilibrium states, basically they are *nonexistent*.

- *Irreversible processes*: In daily life we usually encounter such processes, thermodynamically speaking they are the processes which proceed by themselves until their equilibrium states are reached.

- *Phase transitions*: Phase transition is, easily speaking, an abrupt change of the thermodynamic properties of the system. The most common example is water—it is liquid at room temperature and atmospheric pressure, below the freezing point it becomes solid and when heated above the boiling point it vaporizes. At high pressures water can undergo several addition phase transitions from one solid to another, which makes the ices differ in their crystal structures. We speak of the phase transitions as the change of the entropy of the system (which differs in the phases). There are two kinds of phase transitions, i.e.,

  – First-order phase transition: It is described by an additional heat supply (or release) during the phase transition. The characterizations of the first-order phase transitions are (i) a jump in entropy and (ii) the heat capacity tends to infinity

  – Second-order phase transition: There is no jump in entropy, $\Delta S = 0$ but $\frac{\partial S}{\partial T}$ changes discontinuously at the transition point. We can characterize the second-order phase transition by (i) there is a continous break point in entropy and (ii) there is a finite jump of heat capacity

- *Gibbs free energy*: The Gibbs free energy is defined as

$$G = U - TS + \text{work term} \tag{A.6}$$

  The work term can be PdV or μdN for instance. Practically, G is an indicator of spontaneity of chemical reactions when its change is considered. The reaction is at equilibrium if there is no change in G. If the change is positive then the reaction is spontaneous, and vice versa.

In thermodynamics we have 3 types of systems, namely

- *Isolated systems*: These do not interact in any way with the surroundings. The container has to be impermeable to any form of matter or energy. In this case, the total enery E (mechanical, electrical,...) is a conserved quantity of the system and can be used to characterize the macrostate.

- *Closed systems*: For the closed system, only the exchange of energy with the surroundings is allowed and the energy is therefore not a conserved quantity. The actual energy of the system will fluctuate with the surroudings. Nevertheless, if the closed system is in equilibrium with the surroundings the energy will assume an average value with is related to the temperature of the system.

- *Open systems*: These systems are the most liberal, namely that the exchange of matter and energy with their surroundings are allowed.

- *Ensembles*: In thermodynamics we have 3 types of ensembles which describe the condition for the system:

  – Microcanonical ensemble: When a system is isolated, it is said to have no interaction with its environment, therefore energy and partical number are conserved quantities. Nevertheless, this cannot be realized entirely because any wall in reality is heat-conducting.

  – Canonical ensemble: This is the case for a closed system—so there are fluctuations due to the exchange of energy. We only have the particle number as a conserved quantity.

  – Macrocanonical ensemble: It is more often called *Grand canonical ensemble* In the macrocanonical ensemble we consider the open system in equilibrium with its environment, in which energy and matter of the system can be exchanged with surroundings. In this respect, certain mean values of enery and particle number are established.

Appendix B: Curvature scalar calculation

We often deal with curvature scalars in GR, especially for this work we heavily rely on them since the the curvature scalar of the Ruppeiner metric tells us what statistical model lies beneath the system of interest. To compute them is not always a simple task—it depends mainly on the degree of complication of the algebra and the dimension of the metric tensor. In this thesis we deal with 2 and 3 dimensional metrics and it has proven to be useful to have a formula for obtaining the curvature scalar of the given metric. It is usual that we do have the metric tensor in a diagonal form, therefore we will present here a procedure that helps us reduce such a metric to a simpler form. The line element in 2 dimensions is usually given as

$$ds^2 = A(u,v)\,du^2 + 2B(u,v)\,du\,dv + C(u,v)\,dv^2 \tag{B.1}$$

or

$$g_{ij} = \begin{pmatrix} A(u,v) & B(u,v) \\ B(u,v) & C(u,v) \end{pmatrix} \tag{B.2}$$

We wish to diagonalize it so that the cross term disappears; let λ be a variable such that it satisfies:

$$ds^2 = \frac{1}{A(u,v)\,\lambda^2}\left(\lambda A(u,v)\,du + \lambda B(u,v)\,dv\right)^2 + \left(C(u,v) - \frac{B(u,v)^2}{A(u,v)}\right)dv^2 \tag{B.3}$$

Notice that there is no change in quantity in Eq. B.3 from Eq. (B.1), only symbolically different. The diagonalized metric then reads:

$$ds_\star^2 = L(u,v)\,d\xi^2 + M(u,v)\,dv^2 \tag{B.4}$$

where

$$L(u,v) = \frac{1}{A(u,v)\,\lambda^2} \tag{B.5}$$

$$M(u,v) = \left(C(u,v) - \frac{B(u,v)^2}{A(u,v)}\right) \text{ and} \tag{B.6}$$

$$d\xi = \lambda A(u,v)\,du + \lambda B(u,v)\,dv \tag{B.7}$$

The desired diagonal metric is thus:

$$g_{ij}^\star = \begin{pmatrix} L(u,v) & \\ & M(u,v) \end{pmatrix} \tag{B.8}$$

and we hereafter work with ξ and v instead of u and v. So it should be obvious that the task is indeed to find λ such that the above equations work. However finding the λ is not always trivial. It is a matter of trial and error—some experienced relativists are able to foresee if it is possible to diagonalize the metric at all. Once the diagonalized metric is at hand, one can proceed to compute the curvature scalar of the metric using the following formula:

$$R = \frac{1}{\sqrt{g}}\left[\frac{\partial}{\partial \xi}\left(\frac{1}{\sqrt{g}}\frac{\partial M(u,v)}{\partial \xi}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{g}}\frac{\partial L(u,v)}{\partial v}\right)\right] \tag{B.9}$$

After obtaining the curvature scalar from the above formula; we can then substitute ξ, which is

$$\xi = \int \lambda A(u,v)\,du + \int \lambda B(u,v)\,dv \tag{B.10}$$

so that we get the curvature scalar as a function of u and v. It is worth noting that if λ is not properly chosen (found), the integration for ξ will not be easily to evaluate.

Bibliography

[1] Åman, J., Bengtsson, I and Pidokrajt, N., *Geometry of black hole thermodyanmics*, *General Relativity and Gravitation 35**10***, p. 1733-1743 (2003). (gr-qc/0304015)

[2] Banãdos, M. et.al., *Phys. Rev. Lett.* ***69**, 1849* (1992).

[3] Banãdos, M. et.al., *Phys. Rev. D**48**, 1506* (1993).

[4] Bardeen, J. M., Carter, N. and Hawking, S., *Commun. Math. Phys.* ***31**, 161* (1973).

[5] Bekenstein, J. D., *Phys. Rev. D**7**, 949, 2333* (1973).

[6] Bekenstein, J. D., *Phys. Rev. D**9**, 12* (1974).

[7] Benenson W, et. al., *Handbook of Physics*, Springer (New York 2002).

[8] Caldarelli, M. M., hep-th/9908022.

[9] Callen H. B., *Thermodynamics and introduction to thermostatics, 2nd edition*, John Wiley & Son (Singapore 1985).

[10] Christodoulou, D., *Phys. Rev. Lett.* ***25**, 1596* (1971).

[11] Falcke, H. and Hehl, F. W., *The galactic black hole*, IoP (United Kingdom 2003).

[12] Ferrara, S. et. al., hep-tp/9702103.

[13] Greiner, W. et. al., *Thermodynamics and Statistical Mechanics*, Springer-Verlag (New York 1995).

[14] Hawking, S. W., *Phys. Rev. Lett.* ***26**, 1344* (1971).

[15] Hawking, S. W., *Nature (London)* ***248**, 30* (1974).

[16] Hawking, S. W., *Commun. Math. Phys.* ***43**, 199* (1975).

[17] Hehl, F. W. et. al., *Black holes: Theory and Observation*, Springer (Germany 1998).

[18] Horowitz, G., *Quantum States of Black Holes*, gr-qc/9704072.

[19] Jacobson, T., *Introductory Lectures on Black Hole Thermodynamics*, University of Utrecht.

[20] Kiefer, C., gr-qc/0110070.

[21] Landau, L. D., Lifshitz, E. M., *Statistical Physics*, Pergamon Press (Hungary 1980).

[22] Lynden-Bell, D. and Lynden-Bell, R. M., *Mon. Not. R. astr. Soc.**181**, 405-409* (1977).

[23] Misner, C. W., Thorne, K. S. and Wheeler, J. A., *Gravitation*, W.H. Freeman and Company (San Francisco 1973).

[24] Nieuwenhuizen, Th. M., cond-mat/0103063.

[25] Novikov, I. D. and Frolov, V. P., *Physics of Black holes*, Kluwer Academics Publishing (The Netherlands 1989).

[26] Nulton, J. D. and Salamon, P., *Phys. Rev. A**31**, 2520* (1985).

[27] Penrose, R. and Floyd R. M., *Nature **229**, 177* (1971).

[28] Ruppeiner, G., *Phys. Rev. A**20**, 1608* (1979).

[29] Ruppeiner, G., *Rev. Mod. Phys. **67**, 605* (1995).

[30] Savani, J., *Princeton Weekly Bulletin, April 14, 1997.*

[31] Schutz, B. F., *A first course in general relativity*, Cambride University Press (Cambridge 2001).

[32] Smarr, L., *Phys. Rev. Lett. **30**, 71* (1973).

[33] Strominger, A. and Vafa, C., *Microscopic Origin of the Bekenstein-Hawking Entropy*, gr-qc/9601029

[34] Wald, R. M., *General Relativity*, The University of Chicago Press (Chicago 1984).

[35] Wald, R. M., *Black holes and relativistic stars*, The University of Chicago Press (Chicago 1998).

[36] Wald, R. M., *The Thermodynamics of Black Holes*, Lecture Note, Enrico Fermi Institute and Department of Physics, University of Chicago.

[37] Wald, R. M., http://www.livingreviews.org/Articles/Volume4/2001-6wald/, cited on 19 February 2003.

[38] Weinhold, F., *J. Chem. Phys. **63**, 2479* (1975).

[39] Will, C. M., http://wugrav.wustl.edu/People/CLIFF/exptgravity.html, cited on 17 February 2003.

[40] http://scienceworld.wolfram.com/physics/HeatCapacity.html , cited on 18 March 2003

Index