FYSIKUM
Stockholms Universitet

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Exam, Quantum Mechanics III

Tuesday January 10, 2012
Time: 09:00 – 14:00, Place: Room FR4
Allowed help: Physics Handbook

1. (3 p) Which of the following quantum mechanical statements or expressions are correct, or at least make sense, when interpreted in the "standard" way, and which are not? Give a short explanation for each one:

   (a) If $\Psi(x)$ is a wave function belonging to energy eigenvalue E, then the wave function $\lambda\Psi(x)$ corresponds to the energy eigenvalue λE.

   (b) $\hat{p}|p'\rangle = p'|p'\rangle$

   (c) $(A^\dagger \cdot B^\dagger)^\dagger = B \cdot A$

   (d) The two-particle state with both spins up, $|\alpha\rangle = |\uparrow\uparrow\rangle$ is an entangled state.

   (e) The density matrix

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

   describes a pure ensemble.

   (f) According to Fermi's golden rule, $\Delta_n = \langle n|V|n\rangle$ unless the state is degenerate.

Solution:

(a) Wrong! If $H\Psi = E\Psi$, then $H(\lambda\Psi) = \lambda(H\Psi) = E(\lambda\Psi)$, i.e., $\lambda\Psi$ has the same eigenvalue E.

(b) Right. This is the definition of the eigenket $|p'\rangle$ to the operator $\hat{p}$.

(c) Right! Use that $(B^\dagger)^\dagger = B$ and the rule that a hermitian conjugate of a product of operators is the product of the hermitian conjugated operators in reverse order.

(d) Wrong! It is unentangled. E.g., $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ is an entangled state.

(e) Right! (Check that $\rho^2 = \rho$.)

(f) Wrong! Fermi's golden rule relates to time-dependent perturbations.

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2. (3 p) A particle is in the state $|\alpha\rangle$, with

$$\langle \mathbf{x}|\alpha\rangle = \frac{1}{\sqrt{10}}(x + 3z)f(r),$$

where $r = \sqrt{x^2 + y^2 + z^2}$. What is the probability that a measurement of L_z gives the value $-\hbar$? (Hint: use that the spherical harmonic functions Y_1^m are eigenfunctions of L_z.)

Using the formulas in the formula sheet, we see that

$$\frac{x + iy}{\sqrt{2}} = -r\sqrt{\frac{4\pi}{3}}Y_1^1, \quad \frac{x - iy}{\sqrt{2}} = r\sqrt{\frac{4\pi}{3}}Y_1^{-1}.$$

Thus,

$$x = \frac{r}{\sqrt{2}}\sqrt{\frac{4\pi}{3}}(Y_1^{-1} - Y_1^1),$$

and

$$z = r\sqrt{\frac{4\pi}{3}}Y_1^0.$$

This means that, since the r-dependence is the same, and cancelling also the common square root factor, the wave function is proportional to

$$\frac{1}{\sqrt{2}}Y_1^{-1} - \frac{1}{\sqrt{2}}Y_1^1 + 3Y_1^0$$

and the probability for measuring $-\hbar$ is

$$\frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{2} + 9} = \frac{1}{20}$$

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3. (3 p) A spin 1/2 particle is in the normalized spin state (in a basis where S_z is diagonal)

$$\chi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where β is a real and α is a complex constant. Determine the probabilities for the different possible outcomes of a measurement of S_x and the expectation value $\langle S_x \rangle$.

Solution:

Since β is real, we need only to write α as a complex number, $\alpha = x+iy$. Since χ is normalized we have $|\alpha|^2 + \beta^2 = x^2 + y^2 + \beta^2 = 1$.

$$P\left(S_x = \pm\frac{\hbar}{2}\right) = |\langle S_x = \pm\frac{\hbar}{2}|\chi\rangle|^2 = |\frac{1}{\sqrt{2}}(1, \pm 1)\begin{pmatrix} x+iy \\ \beta \end{pmatrix}|^2,$$

which gives

$$P\left(S_x = \pm\frac{\hbar}{2}\right) = \frac{1}{2}\left((x \pm \beta)^2 + y^2\right) = \frac{1}{2}(1 \pm 2\beta x)$$

(where the normalization condition $\beta^2 + x^2 + y^2 = 1$ has been used). Expectation value:

$$\langle S_x \rangle = +\frac{\hbar}{2}P(S_x = +\frac{\hbar}{2}) - \frac{\hbar}{2}P(S_x = -\frac{\hbar}{2}) = \hbar\beta x = \hbar\beta \mathrm{Re}(\alpha).$$

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4. (3 p) The Hamiltonian for two interacting particles, particle 1 with spin 1/2 and particle 2 with spin 1, in a constant magnetic field along the z-axis, kan be written

$$H = \frac{a}{2}\mathbf{S}_1 \cdot \mathbf{S}_2 + b\hbar\left(S_{1z} + S_{2z}\right),$$

where a and b are real constants. Determine the energy eigenvalues of this system.

Solution:

We use $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, i.e.,

$$S^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2,$$

so the Hamiltonian can be written

$$H = \frac{a}{4}\left(S^2 - S_1^2 - S_2^2\right) + b\hbar S_z.$$

The two spins can be combined to either total spin $s = 1/2$ or $s = 3/2$, and we know that S^2 has eigenvalue $\hbar^2 s(s+1)$, i.e. $3\hbar^2/4$ for $s = 1/2$ and $15\hbar^2/4$ for $s = 3/2$. The eigenvalue of S_1^2 and S_2^2 are $3\hbar^2/4$ and $2\hbar^2$, respectively.

We can thus write down the eigenvalues of the Hamiltonian, i.e. the energy eigenvalues, for state $|s, m_s\rangle =$

$|1/2, -1/2\rangle$: $\frac{\hbar^2}{2}(-a-b)$,

$|1/2, 1/2\rangle$: $\frac{\hbar^2}{2}(-a+b)$,

$|3/2, -3/2\rangle$: $\frac{\hbar^2}{2}(\frac{a}{2}-3b)$,

$|3/2, -1/2\rangle$: $\frac{\hbar^2}{2}(\frac{a}{2}-b)$,

$|3/2, 1/2\rangle$: $\frac{\hbar^2}{2}(\frac{a}{2}+b)$,

$|3/2, 3/2\rangle$: $\frac{\hbar^2}{2}(\frac{a}{2}+3b)$,

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5. (3 p) A system has two non-degenerate states with unperturbed energy E_0 and E_1, with $E_1 > E_0$. At $t = 0$, a perturbation of the form

$$\mathcal{V}(t) = \gamma\left(e^{i\omega t}|E_0\rangle\langle E_1| + e^{-i\omega t}|E_1\rangle\langle E_0|\right)$$

is applied, with γ a real constant. Calculate in lowest order time-dependent perturbation theory the transition rate to the state $|E_1\rangle$ if the system starts in the ground state $|E_0\rangle$, when the applied frequency ω fulfils $\hbar\omega = 2(E_1 - E_0)$.

Solution: Here $\omega_{10} = (E_1 - E_0)/\hbar$, so $\omega = 2\omega_{10}$. Using the formula from the formula sheet, (initial state $i = 0$, final state $n = 1$)

$$|c_1^{(1)}(t)| = |-\frac{i}{\hbar}\int_0^t e^{i\omega_{10}t'}\gamma V_{10}(t')dt'| = \ldots = \frac{2\gamma}{\hbar(\omega-\omega_{10})}|\sin\left(\frac{(\omega-\omega_{10})t}{2}\right)|$$

Thus, since $\omega = 2\omega_{10}$, the probability for transition is

$$|c_1^{(1)}(t)|^2 = \frac{4\gamma^2}{\hbar^2\omega_{10}^2}\sin^2\left(\frac{\omega_{10}t}{2}\right),$$

and the transition rate is

$$w_{0\to1} = \frac{d}{dt}|c_1^{(1)}(t)|^2 = \frac{2\gamma^2}{\hbar^2\omega_{10}} \sin(\omega_{10}t)\,.$$

(This is valid for small t, $\omega t << 1$.)

Some useful formulas

$$\int dx x^{2n} e^{-\lambda x^2} = \frac{\sqrt{\pi}(2n)!}{2^{2n+1} n! \lambda^{n+\frac{1}{2}}}$$

$$\int_0^\infty dx x^{2n+1} e^{-\lambda x^2} = \frac{n!}{2\lambda^{n+1}}$$

$$\int_{-\infty}^{\infty} e^{-a^2t^2} e^{ibt} dt = \sqrt{\frac{\pi}{a^2}} e^{-b^2/(4a^2)}$$

$$J_\pm |j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle$$

For the harmonic oscillator:

$$a|n\rangle = \sqrt{n}|n-1\rangle$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$[a, a^\dagger] = 1$$
$$a^\dagger a|n\rangle = n|n\rangle$$
$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a^\dagger + a\right)$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right)$$
$$\langle l|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1}\right)$$
$$\langle l|\hat{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}} \left(-\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1}\right)$$

Ground state wave function for the one-dimensional harmonic oscillator:

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

Some spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0(\theta) = \sqrt{\frac{3}{4\pi}} \cos\theta$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}} \sin\theta e^{\pm i\phi}$$

$$Y_2^0(\theta) = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

Commutators:

$$[x, F(p)] = i\hbar \frac{\partial}{\partial p} F(p)$$

$$[p, G(x)] = -i\hbar \frac{\partial}{\partial x} G(x)$$

Spin operator for spin-1/2 particles: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Time independent perturbation, non-degenerate case:

$$|n\rangle = |n^{(0)}\rangle + \lambda \sum_{k \neq n} |k^{(0)}\rangle \frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} + \dots$$

$$\Delta_n = E_n - E^{(0)} = \lambda V_{nn} + \lambda^2 \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \dots$$

Time dependent perturbation theory:

$$c_n^{(0)}(t) = \delta_{ni}$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar} \int_{t_0}^{t} \langle n|V_I(t')|i\rangle dt' = -\frac{i}{\hbar} \int_{t_0}^{t} e^{i\omega_{ni}t'} V_{ni}(t') dt'$$

Fermi's Golden Rule:

$$w_{i\to n} = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i)$$

for constant perturbation;

$$w_{i\to n} = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i + \hbar\omega) + \frac{2\pi}{\hbar} |V_{ni}^\dagger|^2 \delta(E_n - E_i - \hbar\omega)$$

for harmonic perturbation.

31. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	…
	M	M	…
m_1 m_2	Coefficients		
m_1 m_2			
⋮			

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\, e^{-im\phi}$$

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m} = (-1)^{m-m'} d^{\,j}_{m,m'} = d^{\,j}_{-m,-m'}$$

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	1/2	4/7	1/35	−2/5
0	3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	1/2	18/35	−3/35	−1/5	1/5
−1	3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	1/2	12/35	−5/14	0	3/10
−2	3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

$$d^{\,1}_{0,0} = \cos\theta \qquad d^{\,1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1} = \frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1} = \frac{1-\cos\theta}{2}$$

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	1	4/7	0	−3/7
0	2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	1	3/7	−1/5	−1/14	3/10
−1	2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	1	8/35	−2/5	1/14	1/10	−1/5
−2	2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$d^{\,3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2} = \Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2} = \Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0} = \Big(\frac{3}{2}\cos^2\theta - \frac{1}{2}\Big)$$

Figure 31.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.