FYSIKUM
Stockholms Universitet

Tentamen, Kvantmekanik III
Exam, Quantum Mechanics III

Tuesday, January 11, 2011
Time: 09:00 – 14:00, Place: Room FR4
Allowed help: Physics Handbook
(and the formulas and tables at the end of this exam)

1. (3 p) Which of the following quantum mechanical statements or expressions are correct, or at least make sense, when interpreted in the "standard" way, and which are not? Give a short explanation for each one:

   (a) If $|x\rangle$ be a position eigenstate, and $|\alpha\rangle$, $|\beta\rangle$ arbitrary states. Then

   $$\langle\alpha|x\rangle\langle x|\beta\rangle = \langle\alpha|\beta\rangle.$$

   (b) $\hat{x}|3\rangle = 3|3\rangle$.

   (c) If $\hat{T}(\vec{a})$ and $\hat{T}(\vec{b})$ represents the translation operator for two diffent translation three-vectors $\vec{a}$ and $\vec{b}$, then

   $$[\hat{T}(\vec{a}), \hat{T}(\vec{b})] \neq 0$$

   in general.

   (d) The commutator between two hermitian operators is hermitian.

   (e) The function $Y_4^5(\theta, \phi)$ is a spherical harmonic $Y_\ell^m(\theta, \phi)$ which represents a state of high orbital angular momentum.

   (f) The interaction picture is defined such that the interactions cause no time-dependence.

Solution:

(a) Wrong! Only if integrated over x, the left hand side becomes equal to the right hand side.

(b) Right! The definition of the eigenket $|x\rangle$ to the operator $\hat{x}$ is that $\hat{x}|x\rangle = x|x\rangle$. Here, $x = 3$.

(c) Wrong! Since $\hat{T}(\vec{a})\hat{T}(\vec{b}) = T(\vec{a}+\vec{b}) = T(\vec{b}+\vec{a}) = T(\vec{b})T(\vec{a})$, the two operators always commute.

(d) Wrong! The commutator changes sign under hermitian conjugation.

(e) Wrong! Y_4^5 does not make sense, since m can at most be equal to l, not larger.

(f) Wrong! The expansion coefficients vary more slowly with time, however.

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2. We consider a spin-1/2 electron in a d-orbital (i.e., with $l = 2$). The relevant part of the hamiltonian can be approximated by

$$H = \frac{\alpha}{\hbar^2}(\vec{J}^2 + J_z^2), \text{with } \alpha > 0,$$

where the total angular momentum is $\vec{J}$ and we have the usual eigenstates $|j, m_j\rangle$. Express the two, degenerate, ground states of this system in terms of the electron's spin and orbital angular momentum, i.e., as linear combinations of $|l, s; m_l, m_s\rangle$. (You may, but do not have to, use the Clebsch-Gordan table attached to this exam.) (3 p)

Solution: The allowed values for j are 5/2 and 3/2. The energy of the system is

$$E = \alpha[j(j+1) + m_j^2].$$

The ground state, i.e that of lowest energy is given by the lowest possible value for this combination of j and m_j, which is for $j = 3/2$ and $m_j = \pm 1/2$. From the CG table we find:

$$|j = 3/2, m_j = 1/2\rangle = \sqrt{\frac{3}{5}}|m_l = 1, m_s = -\frac{1}{2}\rangle - \sqrt{\frac{2}{5}}|m_l = 0, m_s = +\frac{1}{2}\rangle.$$

For the degenerate state with $m_j = -1/2$ we find

$$|j = 3/2, m_j = -1/2\rangle = \sqrt{\frac{2}{5}}|m_l = 0, m_s = -\frac{1}{2}\rangle - \sqrt{\frac{3}{5}}|m_l = -1, m_s = +\frac{1}{2}\rangle.$$

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3. Suppose we have the Hamiltonian

$$H_0 = \frac{p^2}{2m} + V(x),$$

where $V(x) = \frac{1}{2}m\omega^2 x^2$ is a harmonic oscillator potential.

Anharmonic corrections to the Hamiltonian give rise to a term

$$V_1 = \epsilon x^4,$$

with ϵ a small, positive, constant. Compute in first order perturbation theory how this shifts the energy of the state with energy $\frac{5}{2}\hbar\omega$. (3 p)

The state with energy $5\hbar\omega/2$ is the $n = 2$ state. To first order, the energy shift in this state $|2\rangle$ is

$$\Delta E = \langle 2|V_1|2\rangle = \epsilon\langle 2|x^4|2\rangle.$$

Here we use the formula from the formula sheet $x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$, giving

$$\Delta E = \frac{\epsilon\hbar^2}{4m^2\omega^2}\langle 2|\left(a^\dagger + a\right)^4|2\rangle.$$

Here the strategy is to use the commutation relation $[a, a^\dagger] = 1$ repeatedly and note that for a non-vanishing matrix element there has to be equal numbers of factors a and $a^\dagger$. A (rather lengthy) calculation gives the value for the matrix element $\langle n|\left(a^\dagger + a\right)^4|n\rangle = 3(1 + 2n + 2n^2)$, which in the state with $n = 2$ state has the value of 39, and thus

$$\Delta E = \frac{39\epsilon\hbar^2}{4m^2\omega^2}.$$

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4. Consider a particle with charge q moving in a homogeneous, constant magnetic field of magnitude B in the z-direction, $\vec{B} = \nabla \times \vec{A} = B\hat{e}_z$. Let us choose the gauge where $A_z = 0$. We can describe the coupling of the particle to the magnetic field as usual with the minimal coupling prescription, i.e., we replace the free-particle hamiltonian $\vec{p}^2/(2m)$ with

$$\hat{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}(\vec{r})\right)^2 = \frac{1}{2m}\left(\hat{\Pi}_x^2 + \hat{\Pi}_y^2 + p_z^2\right).$$

(a) (1 p) Calculate the commutator $\left[\hat{\Pi}_x, \hat{\Pi}_y\right]$.

(b) (2 p) Compare the hamiltonian with the one-dimensional harmonic oscillator, and use a rescaling of $\hat{\Pi}_x$ to make the commutator in (a) identical to the usual $[\hat{x}, \hat{p}] = i\hbar$. Then use this to calculate the energy eigenvalues of $\hat{H}$.

Solution: (a) Using $\vec{p} = -i\hbar\nabla$, we find $\left[\hat{\Pi}_x, \hat{\Pi}_y\right] = \frac{i\hbar q}{c}\left(\partial_x A_y - \partial_y A_x\right) = \frac{i\hbar q}{c} B$.

(b) The commutation relation in (a) has $i\hbar \cdot \frac{qB}{c} \equiv i\hbar C$, with C a constant instead of $i\hbar$ on the right hand side. Let us put $C = qB/c = m\omega$, which defines $\omega = qB/(mc)$. If we rescale, $\hat{Y} = \hat{\Pi}_x / C$, the hamiltonian becomes

$$\hat{H} = \frac{\hat{\Pi}_y^2}{2m} + \frac{1}{2} m\omega^2 \hat{Y}^2 + \frac{\hat{p}_z^2}{2m},$$

with $\omega = qB/mc$.

Here we recognize the one-dimensional harmonic oscillator in the first two terms, along with free motion along the z-direction. Since $\left[\hat{Y}, \hat{\Pi}_y\right] = i\hbar$, we can use the usual methods of the harmonic oscillator with creation and annihilation operators to immediately find the eigenvalues:

$$E_n(p_z) = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{p_z^2}{2m},$$

with $\omega = qB/mc$, and p_z an arbitrary, real number.

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5. (3 p) A particle of charge Q and mass m is in a one-dimensional harmonic oscillator potential, with fundamental frequency ω. It is subjected to a time-dependent electric field, in the direction along the displacement of the oscillator, obtained from by the perturbing potential

$$V_1(x,t) = \frac{Q\epsilon x}{\sqrt{\pi}\tau} e^{-\left(\frac{t}{\tau}\right)^2},$$

with ϵ and τ real positive constants. If the oscillator starts in the ground state when $t \to -\infty$, what is the probability that it will be found in an excited state as $t \to +\infty$?

Solution: We use the formula for time-dependent perturbations given in the formula sheet (with starting point t_0 replaced by $t = -\infty$ and the initial state $|i\rangle$ being the ground state $|0\rangle$)

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{-\infty}^{t} \langle n|V_I(t')|0\rangle dt' = -\frac{i}{\hbar}\int_{-\infty}^{t} e^{i\omega_{nt}t'} V_{n0}(t')dt'.$$

Here the matrix element

$$V_{n0}(t') = \langle n|\frac{Q\epsilon x}{\sqrt{\pi}\tau} e^{-\left(\frac{t'}{\tau}\right)^2}|0\rangle.$$

After using $\langle n|x|0\rangle = \delta_{n,1}$ we find

$$\lim_{t\to\infty} c_1^{(1)}(t) = \sqrt{\frac{1}{2\hbar m\omega}}\frac{Q\epsilon}{\sqrt{\pi}\tau}\int_{-\infty}^{+\infty} e^{i\omega t'} e^{-\left(\frac{t'}{\tau}\right)^2} dt'.$$

A look in the table of integrals gives

$$\int_{-\infty}^{+\infty} e^{i\omega t'} e^{-\left(\frac{t'}{\tau}\right)^2} dt' = \sqrt{\pi}\tau e^{-\frac{1}{4}\omega^2\tau^2}$$

and, finally, the limiting probability

$$\lim_{t\to+\infty} |c_1^{(1)}(t)|^2 = \frac{Q^2\epsilon^2}{2\hbar m\omega} e^{-\frac{1}{2}\omega^2\tau^2}.$$

GOOD LUCK!

Some useful formulas

$$\int dx x^{2n} e^{-\lambda x^2} = \frac{\sqrt{\pi}(2n)!}{2^{2n+1} n! \lambda^{n+\frac{1}{2}}}$$

$$\int_0^\infty dx x^{2n+1} e^{-\lambda x^2} = \frac{n!}{2\lambda^{n+1}}$$

$$\int_{-\infty}^{\infty} e^{-a^2t^2} e^{ibt} dt = \sqrt{\frac{\pi}{a^2}} e^{-b^2/(4a^2)}$$

$$J_\pm |j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m+1)} |j, m \pm 1\rangle$$

For the harmonic oscillator:

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1}|n+1\rangle$$

$$[a, a^\dagger] = 1$$

$$a^\dagger a|n\rangle = n|n\rangle$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a^\dagger + a\right)$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right)$$

$$\langle l|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1}\right)$$

$$\langle l|\hat{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}} \left(-\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1}\right)$$

Ground state wave function for the one-dimensional harmonic oscillator:

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

Some spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0(\theta) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0(\theta) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Commutators:

$$[x, F(p)] = i\hbar\frac{\partial}{\partial p}F(p)$$

$$[p, G(x)] = -i\hbar\frac{\partial}{\partial x}G(x)$$

Spin operator for spin-1/2 particles: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$
Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Time independent perturbation, non-degenerate case:

$$|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} + \dots$$

$$\Delta_n = E_n - E^{(0)} = \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \dots$$

Time dependent perturbation theory:

$$c_n^{(0)}(t) = \delta_{ni}$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = -\frac{i}{\hbar}\int_{t_0}^{t}e^{i\omega_{ni}t'}V_{ni}(t')dt'$$

Fermi's Golden Rule:

$$w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i)$$

for constant perturbation;

$$w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i + \hbar\omega) + \frac{2\pi}{\hbar}|V_{ni}^\dagger|^2\delta(E_n - E_i - \hbar\omega)$$

for harmonic perturbation.