FYSIKUM
Stockholms Universitet

Tentamen, Kvantmekanik III
Exam, Quantum Mechanics III

Tuesday, January 12, 2010
Time: 09:00 – 15:00, Place: Room FR4
Allowed help: Physics Handbook
(and the formulas and tables at the end of this exam)

1. (3 p) Which of the following quantum mechanical statements or expressions are correct, or at least make sense, when interpreted in the "standard" way, and which are not? Give a short explanation for each one:

   (a) If $|1\rangle$ and $|2\rangle$ are eigenstates corresponding to different energy eigenvalues E_1 and E_2, respectively, then $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$ is also an energy eigenstate.

   (b) $\hat{x}|y\rangle = x|y\rangle$

   (c) $(A \cdot B \cdot C)^\dagger = B^\dagger \cdot A^\dagger \cdot C^\dagger - C^\dagger \cdot A^\dagger \cdot B^\dagger$

   (d) $\sum_a |a\rangle\langle a|\beta\rangle = |\beta\rangle$

   (e) e^{iU} is unitary, if U is hermitian.

   (f) The Heisenberg operator $\hat{x}_H = e^{\frac{i}{\hbar}Ht}\hat{x}e^{-\frac{i}{\hbar}Ht}$ does not depend on time if the Hamiltonian H is time-independent.

Solution:

(a) Wrong! If $H|i\rangle = E_i|i\rangle$, then $H(|1\rangle + |2\rangle) = E_1|1\rangle + E_2|2\rangle$, which is not equal to $(E_1 + E_2)(|1\rangle + |2\rangle)$.

(b) Wrong! The definition of the eigenket $|y\rangle$ to the operator $\hat{x}$ is that $\hat{x}|y\rangle = y|y\rangle$.

(c) Wrong! The right expression would be $(A \cdot B \cdot C)^\dagger = C^\dagger \cdot B^\dagger \cdot A^\dagger$ (use the relation $(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger$ on $(A \cdot B)$ and C).

(d) Right. The "most important formula" of the course.

(e) Right. An operator U is hermitian if $U^\dagger = U$ (don't be fooled by the fact that we call it U here). Here, $(e^{iU})^\dagger = e^{-iU^\dagger} = e^{-iU}$, and $e^{iU}e^{-iU} = 1$.

(f) Wrong! In the Heisenberg picture, operators are time-dependent. (Check the simple example of a free particle in one dimension. One finds $x_H(t) = (p/m)t + x_H(0)$.)

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2. Unpolarized spin-1 particles are sent through a Stern-Gerlach apparatus which analyses the projections of the spin in the direction

$$\hat{n} = \frac{1}{\sqrt{2}}(\hat{e}_x + \hat{e}_z).$$

(a) (0.5 p) How many outgoing beams will be generated by the Stern-Gerlach apparatus?

(b) (2.5 p) A second SG apparatus measuring the spin in direction $\hat{e}_z$ takes the beam with $J_{\hat{n}} = 0$. What is the probability that $J_z = 0$ is measured? (Hint: Write $J_{\hat{n}} = \vec{J} \cdot \hat{n}$, and use $J_x = (J_+ + J_-)/2$.)

Solution:

(a) The beam will split in three, corresponding to the eigenvalues of J_z being 0 and $\pm\hbar$.

(b) The eigenvalue equation is $J_{\hat{n}}|j_n, 0\rangle = 0 \cdot |j_n, 0\rangle = 0$. Since the second SG apparatus measures in the z-direction, we should expand the $J_{\hat{n}} = 0$ state in eigenstates of J_z:

$$|j_n, 0\rangle = c_1|j_z, +1\rangle + c_0|j_z, 0\rangle + c_{-1}|j_z, -1\rangle.$$

Insert this in the first equation to find

$$J_{\hat{n}}|j_n, 0\rangle = 0 = \frac{1}{\sqrt{2}}\left((J_+ + J_-)/2 + J_z\right)\left(c_1|j_z, +1\rangle + c_0|j_z, 0\rangle + c_{-1}|j_z, -1\rangle\right) = \dots$$
$$= \frac{\hbar}{\sqrt{2}}\left[\left(\frac{c_0}{\sqrt{2}} + c_1\right)|j_z, +1\rangle \left(\frac{c_1 + c_{-1}}{\sqrt{2}}\right)|j_z, 0\rangle \left(\frac{c_0}{\sqrt{2}} - c_{-1}\right)|j_z, -1\rangle\right].$$

This gives $c_0 = \sqrt{2}c_{-1} = -\sqrt{2}c_1$, and one finds the probability

$$P(J_z = 0) = 2/(2 + 1 + 1) = 1/2.$$

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3. Compute the energy shift of the one-dimensional harmonic oscillator, to first order in perturbation theory, for the perturbation

   (a) (1p) $V = \lambda_1 x^4$, for the ground state $|0\rangle$,

   (b) (2p) $V = \lambda_2 p^4$, for the state $|n\rangle$,

Solution: For $n = 0$ in (a), the easiest is to use the explicit form of the harmonic oscillator wave function $\Psi_0(x) = \langle x|0\rangle$ given in the formula sheet and integrate $\int \Psi_0(x)x^4\Psi_0(x)dx$. One finds $V_{00} = \frac{3\lambda_1\hbar^2}{4m^2\omega^2}$. For general n both cases can be solved by essentially the same calculation:

$$V_{nn} \propto \langle n| \left(a \pm a^\dagger\right)^4 |n\rangle.$$

Only terms with a product of two a's and two $a^\dagger$'s in arbitrary order will give a non-vanishing contribution, due to the orthogonality of states with different n. By using $[a, a^\dagger] = 1$ one may successively replace all the operators by $a^\dagger a = n$ when acting on the state $|n\rangle$. Example: $\langle n|aa^\dagger aa^\dagger|n\rangle = \langle n|(a^\dagger a+1)(a^\dagger a+1)|n\rangle = (n+1)^2$. One finds $V_{nn}/\lambda_i \propto \frac{3(1+2n+2n^2)\hbar^2}{4m^2\omega^2}$ in both (a) and (b).

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4. (3 p) Estimate the ground state energy of a particle of mass m, moving in a linear potential,

$$V(x) = \kappa \cdot |x|, \quad -\infty < x < \infty, \quad \kappa > 0$$

using the variational trial wave function $\Psi(x) = e^{-\lambda x^2}$, where λ is to be varied.

Solution: We use the relation

$$\bar{H}(\lambda) = \frac{\langle\tilde{0}|H|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \geq E_0,$$

and minimize by demanding $d\bar{H}(\lambda)/d\lambda = 0$ The found value $\lambda_{\min}$ is then inserted into $\bar{H}$ to find the best estimate of the ground state energy E_0. We find, using the integration tables in the formula sheet,

$$\langle\tilde{0}|\tilde{0}\rangle = \int_{-\infty}^{\infty} e^{-2\lambda x^2}dx = 2\int_0^{\infty} e^{-2\lambda x^2}dx = \sqrt{\frac{\pi}{2\lambda}},$$

$$\langle\tilde{0}|H|\tilde{0}\rangle = 2\int_0^\infty e^{-\lambda x^2}\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}+\kappa x\right)e^{-\lambda x^2}dx = \ldots = \frac{\hbar^2}{2m}\sqrt{\frac{\pi\lambda}{2}}+\frac{\kappa}{2\lambda}.$$

Thus,

$$\bar{H}(\lambda) = \frac{\langle\tilde{0}|H|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = \frac{\hbar^2\lambda}{2m}+\frac{\kappa}{\sqrt{2\pi\lambda}}$$

Solving $d\bar{H}(\lambda)/d\lambda = 0$ gives

$$\lambda = \left(\frac{m\kappa}{\sqrt{2\pi}\hbar^2}\right)^{\frac{2}{3}},$$

and that inserted in $\bar{H}$ gives the estimate (the smallest upper bound)

$$E_0 \sim 3\left(\frac{\hbar^2}{2m}\right)^{\frac{1}{3}}\left(\frac{\kappa}{2\sqrt{2\pi}}\right)^{\frac{2}{3}}.$$

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5. (3 p) A free particle of spin-1 is at rest in a magnetic field B_z parallel to the z-axis, so that

$$H_0 = -\kappa B_z S_z,$$

where S_z is the projection of the spin operator in the z-direction. A harmonic perturbation with frequency $\omega = \kappa B_z$ is applied in the x-direction for a short time (one period of oscillation only), so that the weak perturbation is

$$V(t) = -\kappa B_x \sin(\omega t) S_x, \text{ for } 0 < t < T = \frac{2\pi}{\omega}.$$

Calculate, using first order time-dependent perturbation theory, the resulting state vector $|\alpha, t_0 = 0; t\rangle$, for $t > T$, if the initial state is $|m = 0\rangle$.

Solution: The unperturbed Hamiltonian has eigenstates $|m\rangle$, $m = -1, 0, 1$ with eigenvalues $E_m^{(0)} = -\kappa B_z \hbar m = \hbar\omega_m$. The initial state is $|0\rangle$, and $|\alpha, t_0 = 0; t\rangle = \sum_m c_m(t) e^{-i\omega_m t}|m\rangle$. Perturbation theory: $c_m^{(0)} = \delta_{m0}$,

$$c_m^{(1)} = \frac{-i}{\hbar}\int_0^T \langle m|V_I(t')|0\rangle dt'$$

$$= \frac{-i}{\hbar}\int_0^T \langle m|e^{i\omega_{m0}t'}V_{m0}(t')|0\rangle dt'$$
$$= \frac{i\kappa B_x}{\hbar}\langle m|S_x|0\rangle \int_0^T \sin(\omega t)e^{i\omega_{m0}t}dt,$$

with $\omega_{m0} = \omega_m - \omega_0 = \omega_m = -\omega m$. Here, $\langle m|S_x|0\rangle = \frac{1}{2}\langle m|(S_+ + S_-)|0\rangle = \frac{\hbar}{\sqrt{2}}(\delta_{m,1}+\delta_{m,-1})$, and the integral over one period is computed to be $\frac{mT}{2i} = \frac{\pi m}{i\kappa B_z}$.

Thus, $c_m = \delta_{m0} + \frac{B_x\pi}{B_z\sqrt{2}}m(\delta_{m,1} + \delta_{m,-1})$, and $|\alpha, t_0 = 0; t > T\rangle = |0\rangle + \frac{\pi B_x}{\sqrt{2}B_z}\left(e^{i\omega t}|1\rangle - e^{-i\omega t}|-1\rangle\right)$. To obtain a normalized state, one should finally divide by the factor $\sqrt{1+\pi^2 B_x^2/B_z^2}$.

GOOD LUCK!

Some useful formulas

$$\int dx x^{2n} e^{-\lambda x^2} = \frac{\sqrt{\pi}(2n)!}{2^{2n+1} n! \lambda^{n+\frac{1}{2}}}$$

$$\int_0^\infty dx x^{2n+1} e^{-\lambda x^2} = \frac{n!}{2\lambda^{n+1}}$$

$$J_\pm |j,m\rangle = \hbar\sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle$$

For the harmonic oscillator:

$$a|n\rangle = \sqrt{n}|n-1\rangle$$
$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$[a, a^\dagger] = 1$$
$$a^\dagger a|n\rangle = n|n\rangle$$
$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right)$$

$$p = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^\dagger - a\right)$$
$$\langle l|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1}\right)$$
$$\langle l|\hat{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1}\right)$$

Ground state wave function for the one-dimensional harmonic oscillator:

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

Some spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0(\theta) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^{\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\phi}$$

$$Y_2^0(\theta) = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

Commutators:

$$[x, F(p)] = i\hbar\frac{\partial}{\partial p}F(p)$$

$$[p, G(x)] = -i\hbar\frac{\partial}{\partial x}G(x)$$

Spin operator for spin-1/2 particles: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$
Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Time independent perturbation, non-degenerate case:

$$|n\rangle = |n^{(0)}\rangle + \lambda\sum_{k\neq n}|k^{(0)}\rangle\frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} + \ldots$$

$$\Delta_n = E_n - E^{(0)} = \lambda V_{nn} + \lambda^2\sum_{k\neq n}\frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \ldots$$

Time dependent perturbation theory:

$$c_n^{(0)}(t) = \delta_{ni}$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}\langle n|V_I(t')|i\rangle dt' = -\frac{i}{\hbar}\int_{t_0}^{t} e^{i\omega_{nt}t'}V_{ni}(t')dt'$$

Fermi's Golden Rule:

$$w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i)$$

for constant perturbation;

$$w_{i\to n} = \frac{2\pi}{\hbar}|V_{ni}|^2\delta(E_n - E_i - \hbar\omega) + \frac{2\pi}{\hbar}|V_{ni}^\dagger|^2\delta(E_n - E_i + \hbar\omega)$$

for harmonic perturbation.