

FYSIKUM
Stockholms Universitet

Tentamen, Kvantmekanik III Exam, Quantum Mechanics III

Tuesday January 13, 2009
Time: 09:00 – 15:00, Place: Room FR4
Allowed help: Physics Handbook

1. (3 p) Which of the following quantum mechanical expressions are correct, or at least make sense, when interpreted in the “standard” way, and which do not? Give a short explanation for each one:
- (a) $\psi_\alpha(x) = \langle x|\alpha\rangle$
 - (b) $\hat{x}|y\rangle = x|y\rangle$
 - (c) $(A^\dagger \cdot B^\dagger)^\dagger = A \cdot B$
 - (d) $c\langle\alpha|\beta\rangle = \langle c|\alpha\rangle|\beta\rangle$
 - (e) e^A is unitary, if A is hermitian.
 - (f) $\rho(\mathbf{r}, t) = |\psi^*(\mathbf{r}, t)|^2$

Solution:

- (a) OK. The definition of the wave function, i.e. the state vector in the x -representation.
- (b) Not OK. The correct relation should read $\hat{x}|y\rangle = y|y\rangle$, since by \hat{x} we mean the position operator.
- (c) Not OK, should be $B \cdot A$ on the right hand side, since $(A^\dagger)^\dagger = A$ and same for B , and $(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger$.
- (d) Not OK. Left hand side is a (complex) number, right hand side is a state vector.
- (e) Not correct, a factor of $\pm i$ has to be added in the exponential to make $e^{\pm iA}$ unitary.
- (f) OK. The definition of the probability density. (Note that $|a|^2 = |a^*|^2$ for all complex numbers a .)
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2. (3 p) A particle is in the state $|\alpha\rangle$, with

$$\langle \mathbf{x} | \alpha \rangle = \frac{1}{\sqrt{2}}(x + iy + 2iz)f(r),$$

where $r = \sqrt{x^2 + y^2 + z^2}$. What is the probability that a measurement of L_z gives the value $+\hbar$? (Hint: use that the spherical harmonic functions Y_1^m are eigenfunctions of L_z .)

Using the formulas in the formula sheet, we see that

$$\frac{x + iy}{\sqrt{2}} = -r\sqrt{\frac{4\pi}{3}}Y_1^1$$

and

$$z = r\sqrt{\frac{4\pi}{3}}Y_1^0$$

This means that, since the r -dependence is the same, and cancelling also the common square root factor, the wave function is proportional to

$$-Y_1^1 + \sqrt{2}iY_1^0$$

and the probability for measuring $+\hbar$ is

$$\frac{|-1|^2}{|-1|^2 + |i\sqrt{2}|^2} = \frac{1}{3}$$

3. (3 p) An electron (spin-1/2) is in a state $|n\ell m_\ell\rangle$ where n labels the energy eigenvalue, and where the orbital angular momentum is ℓ and its z -component m_ℓ . Use a description in terms of the total angular momentum i.e., $\mathbf{J} = \mathbf{L} + \mathbf{S}$, and consider the state with $j = \ell + 1/2$ and $m = m_\ell - 1/2$. Calculate the probability that a measurement of the z -component S_z of the spin gives the value $+\hbar/2$ for the cases $\ell = 1$ and $\ell = 2$, respectively.
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As was stated on the blackboard at the exam, we only need to consider the case where m_ℓ is maximal, i.e., $m_\ell = \ell$ (this was granted, to avoid

extensive calculations). We note that the value of n has no importance of the problem, so we only have to know how to couple ℓ and $s = 1/2$. The coefficients we need are the well-known Clebsch-Gordan coefficients. The easiest way to solve the problem is thus to use the table of Clebsch-Gordan coefficients. For $\ell = 1$, we have $j = 3/2$ and $m = 1/2$. In the table for $1 \times 1/2$ we find for this case the coefficient $2/3$. Remembering that what is given in the table is really the square of the coefficient, we immediately find the probability $p = 2/3$. Similarly, for $\ell = 2$, we use the $2 \times 1/2$ table and find, for $j = 5/2$ and $m = 3/2$, $p = 4/5$.

4. (3 p) The Hamiltonian for two interacting spin- $1/2$ particles in a constant magnetic field along the z -axis can be written

$$H = \frac{a}{4} \mathbf{S}_1 \cdot \mathbf{S}_2 + b\hbar (S_{1z} + S_{2z}),$$

where a and b are constants. Determine the energy eigenvalues of this system.

We use $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$, i.e.,

$$S^2 = S_1^2 + S_2^2 + 2\mathbf{S}_1 \cdot \mathbf{S}_2,$$

so the Hamiltonian can be written

$$H = \frac{a}{8} (S^2 - S_1^2 - S_2^2) + b\hbar S_z.$$

The two spins can be combined to either total spin $s = 0$ or $s = 1$, and we know that S^2 has eigenvalue $\hbar^2 s(s+1)$, i.e. 0 for $s = 0$ and $2\hbar^2$ for $s = 1$. The eigenvalue of S_1^2 and S_2^2 is $3\hbar^2/4$ (being spin- $1/2$ particles).

We can thus write down the eigenvalues of the Hamiltonian, i.e. the energy eigenvalues, for state $|s, m_s\rangle =$

$$|0, 0\rangle: \frac{-3a\hbar^2}{16},$$

$$|1, 0\rangle: \frac{a\hbar^2}{16},$$

$$|1, 1\rangle: \frac{a\hbar^2}{16} + b\hbar^2,$$

$$|1, -1\rangle: \frac{a\hbar^2}{16} - b\hbar^2.$$

5. (3 p) A harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

is in its ground state. Then, for $t \geq 0$ a perturbation of the form

$$V(x, t) = \delta m\omega^2 x^2 e^{-t/\tau}$$

is applied, with δ and τ real and positive. Using first-order time dependent perturbation theory, compute the probability that the harmonic oscillator is not in the ground state when $t \rightarrow \infty$.

According to the formula for time-dependent perturbation theory,

$$c_n^{(1)}(t) = -\frac{i}{\hbar} \int_0^t \langle n | V_I(t') | 0 \rangle dt' = -\frac{i}{\hbar} \int_0^t e^{i\omega_{n0}t'} V_{n0}(t') dt'$$

with $\hbar\omega_{n0} = E_n - E_0 = n\hbar\omega$. Here the t' - and x -dependence can be separated, so that

$$V_{n0}(t') = \langle n | \delta m\omega^2 x^2 e^{-t'/\tau} | 0 \rangle = \delta m\omega^2 e^{-t'/\tau} \langle n | x^2 | 0 \rangle$$

Using the formula

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$$

we see that

$$\begin{aligned} V_{n0}(t') &= \delta m\omega^2 e^{-t'/\tau} \langle n | x^2 | 0 \rangle = \frac{\hbar\omega\delta e^{-t'/\tau}}{2} \langle n | (a + a^\dagger)^2 | 0 \rangle = \\ &= \frac{\hbar\omega\delta e^{-t'/\tau}}{2} \langle n | (|0\rangle + \sqrt{2}|2\rangle) \end{aligned}$$

We are only interested in the part that is not the ground state ($n = 0$), so only $n = 2$ contributes, with

$$\lim_{t \rightarrow \infty} c_2^{(1)}(t) = -\frac{i}{\hbar} \int_0^\infty e^{i2\omega t'} \frac{\hbar\omega\delta}{\sqrt{2}} e^{-t'/\tau} dt' = \frac{-i\omega\delta}{\sqrt{2}(2i\omega - \frac{1}{\tau})}$$

Thus, the transition probability to first order as $t \rightarrow \infty$ is

$$|c_2^{(1)}|^2 = \frac{\omega^2 \delta^2}{8\omega^2 + \frac{2}{\tau^2}}$$

Some useful formulas

$$\int_0^\infty dx x^{2n} e^{-\lambda x^2} = \frac{\sqrt{\pi}(2n)!}{2^{2n+1} n! \lambda^{n+\frac{1}{2}}}$$

$$\int_0^\infty dx x^{2n+1} e^{-\lambda x^2} = \frac{n!}{2\lambda^{n+1}}$$

$$J_\pm |j, m\rangle = \hbar \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle$$

For the harmonic oscillator:

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

$$[a, a^\dagger] = 1$$

$$a^\dagger a|n\rangle = n|n\rangle$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a^\dagger + a)$$

Note: There was a misprint in the sheet given out at the exam, making x a factor of 2 too big.

$$p = i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a)$$

$$\langle l|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1})$$

$$\langle l|\hat{p}|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}} (-\sqrt{n}\delta_{l,n-1} + \sqrt{n+1}\delta_{l,n+1})$$

Ground state wave function for the harmonic oscillator:

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

Some spherical harmonics:

$$Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$Y_1^0(\theta) = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_1^{\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}$$

$$Y_2^0(\theta) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$$

$$L_z = -i\hbar \frac{\partial}{\partial \phi}$$

Commutators:

$$[x, F(p)] = i\hbar \frac{\partial}{\partial p} F(p)$$

$$[p, G(x)] = -i\hbar \frac{\partial}{\partial x} G(x)$$

Spin operator for spin-1/2 particles: $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$

Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Time independent perturbation, non-degenerate case:

$$|n\rangle = |n^{(0)}\rangle + \lambda \sum_{k \neq n} |k^{(0)}\rangle \frac{V_{kn}}{E_n^{(0)} - E_k^{(0)}} + \dots$$

$$\Delta_n = E_n - E^{(0)} = \lambda V_{nn} + \lambda^2 \sum_{k \neq n} \frac{|V_{nk}|^2}{E_n^{(0)} - E_k^{(0)}} + \dots$$

Time dependent perturbation theory:

Fermi's Golden Rule

$$w_{i \rightarrow n} = \frac{2\pi}{\hbar} |V_{ni}|^2 \delta(E_n - E_i)$$

$$c_n^{(0)}(t) = \delta_{ni}$$

$$c^{(1)}(t) = -\frac{i}{\hbar} \int_{t_0}^t \langle n | V_I(t') | i \rangle dt' = -\frac{i}{\hbar} \int_{t_0}^t e^{i\omega_{nt}t'} V_{ni}(t') dt'$$