

STATISTICAL PHYSICS

WINTER 2012

There will be lectures following Daniel Schroeder's book on Mondays, Tuesdays, Wednesdays, and Fridays—the course will be an example of high-velocity teaching—in FB41.

We go through the chapters of the book according the following outline:

January 16, 17 Introduction (mostly thermodynamics)

January 18, 20 Schroeder chapter 6

January 23 Exercise class

January 24, 25 Schroeder chapter 7

January 27 Exercise class

January 30, February 1 Schroeder chapter 7

February 3 Schroeder chapter 5.3

February 6 Exercise class

February 7 Schroeder chapters 5.3, 8.2

February 8 Exercise class

February 10 Schroeder 7.6 + exercise class

February 13 Ising model demonstration (Kate Blanchfield)

February 15 Summary of the course

February 17 Exam.

All lectures are in FB41, at 13.15 on Mondays, 10.15 on Tuesdays, 8.15 on Wednesdays, and 10.15 on Fridays. The exam is in FR4. Problem solving is an

essential part of the lectures; the precise problems to be discussed will be given below as we proceed.

There are seven possible grades, “excellent”, “very good”, “good”, “satisfactory”, “enough”, “not enough”, and “not even close”.

“Kursforum”: Alvin Gavel, Anders Lundkvist, and David Andersson.

My office is A5:1057. I am happy to discuss statistical physics with visitors.

PROBLEMS TO BE DISCUSSED

January 23:

Suppose

$$S = S(U, V) = N \ln \left[\frac{V - bN}{N} \left(\frac{U + \frac{aN^2}{V}}{N} \right)^{\frac{3}{2}} \right] + c_0 N . \quad (1)$$

Work out the equation of state (a relation between pressure, temperature, and volume). What is this?

Schroeder, exercises 6.19, 6.22, 6.32.

January 27: 6.43, 6.44, 6.50, 7.22, 7.23.

February 6: 7.6, 7.15, 7.41, 7.44, 7.45.

February 8: 5.47, 5.48, 5.51, 5.54, 5.55 a-e

February 10: An old exam (see below).

A MESSAGE FROM OUR SPONSOR

Here are some useful facts from analysis. The first fact is that

$$\int_{-\infty}^{\infty} dx e^{-x^2} = \sqrt{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-x^2-y^2}} = \sqrt{2\pi \int_0^{\infty} dr r e^{-r^2}} = \sqrt{\pi} . \quad (2)$$

This is known as the Gaussian integral. By the way the indefinite integral

$$\operatorname{erf}(x) = \int_{-\infty}^x dx e^{-x^2} \quad (3)$$

defines what is known as the error function, which is very important in statistics and used to be tabulated in books on that subject.

In a little more generality we can compute

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} . \quad (4)$$

We can then compute the moments of the normal distribution. The odd moments vanish, while

$$\langle x^{2n} \rangle = \frac{\int_{-\infty}^{\infty} dx x^{2n} e^{-x^2/2}}{\int_{-\infty}^{\infty} dx e^{-x^2/2}} = \frac{(2n)!}{2^n n!} = (2n-1)!! . \quad (5)$$

Note that the double factorial function $(2n-1)!! = (2n-1) \cdot (2n-3) \cdot \dots \cdot 1$ is the number of ways in which $2n$ objects can be collected into n pairs. This is related to Wick contractions in quantum field theory. The formula can be proved by induction in one of two ways; either using partial integration, or by applying the operator

$$-2 \frac{d}{da} \quad (6)$$

to eq. (4).

We will also be interested in Euler's Gamma-function

$$\Gamma(x) = \int_0^{\infty} dt t^{x-1} e^{-t} . \quad (7)$$

By the way this is the Mellin transform of the function e^{-t} . Again using integration by parts we see that

$$\Gamma(x + 1) = x\Gamma(x) . \quad (8)$$

We also observe that

$$\Gamma(1) = 1 , \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} . \quad (9)$$

The second equality is derived by setting $t = s^2$, and then evaluating the resulting Gaussian integral. Armed with this information we deduce that

$$\Gamma(n) = (n - 1)! , \quad (10)$$

where n is an integer. Thus the Gamma-function can be thought of as the extension to arbitrary complex arguments of the factorial function. We also see that

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n - 1)!!}{2^n} \sqrt{\pi} , \quad (11)$$

where the double factorial function appears again.

The factorial function and its behaviour when n is large is of considerable interest in statistical physics. A quick mnemonic for its behaviour at large n is the observation that

$$\ln n! = \sum_{k=1}^n \ln k \approx \int_0^n \ln x = n \ln n - n , \quad (12)$$

where integration by parts was used in the second step. A more accurate version is Stirling's formula

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n . \quad (13)$$

This approximation is surprisingly good. The Stirling series tells us that

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \dots\right) . \quad (14)$$

Therefore the approximation is correct to better than one percent already for $n = 10$, while the rougher approximation for the logarithm is better than one percent already for $n = 100$.

Possibly the most famous function in mathematics is Riemann's zeta-function. Provided that $\text{Re}(z) > 1$ (to ensure that the series converges) it is given by

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} . \quad (15)$$

Like the Gamma-function it can be analytically continued to the entire complex plane. Simple expressions for its value when z equals an even integer are known. Two special values are

$$\zeta(2) = \frac{\pi^2}{6} , \quad \zeta(4) = \frac{\pi^4}{90} . \quad (16)$$

They are used in statistical physics. Also the reciprocal of the first—for your information—gives the probability that two integers selected at random are relatively prime. The zeta-function vanishes when z is an even negative integer. Riemann conjectured that all remaining zeroes lie on the line $\operatorname{Re}(z) = 1/2$. A proof—or a disproof—of this conjecture would affect large areas of mathematics and mathematical physics (including random matrix theory and quantum chaos).

Given the importance of the Riemann hypothesis it is perhaps allowable to write down, without proof, the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin \frac{\pi z}{2} \Gamma(1-z) \zeta(1-z) . \quad (17)$$

The existence of the trivial zeroes at $z = -2n$ follows from the fact that the analytically continued Gamma-function has poles at every non-positive integer (that is, it equals ∞ there). The functional equation also suggests that there is something very special about the real value $1/2$.

Tentamen i Statistisk Fysik I den 25 februari 2011, i FB41, under tiden 9.00-14.00. Lärare: Ingemar Bengtsson. Hjälpmedel: Penna, suddgummi och linjal. Bedömning: Uppgift 1-2 3 poäng/uppgift, 3-5 4 poäng/uppgift. Betyg: 0-3 = F, 4-6 = Fx, 7-9 = E, 10-12 = D, 13-14 = C, 15-16 = B, 17-18 = A.

1. Boltzmann's postulate is that $S = k \ln \Omega$, where Ω is the accessible phase space volume. Use it to show that the entropy of the ideal monoatomic gas depends on volume and energy according to

$$S = kN \ln [VU^{\frac{3}{2}}] + f(N) , \quad (18)$$

where f is some function of N that you need not compute.

2. The entropy of the photon gas is

$$S = \frac{4\sqrt{2}}{3} \left(\frac{\sigma}{c}\right)^{\frac{1}{4}} V^{\frac{1}{4}} U^{\frac{3}{4}} . \quad (19)$$

Compute the entropy S and the energy U as functions of T and V .

3. At low temperatures the specific heat of a metal behaves like

$$C_V = k_1 T + k_2 T^3 , \quad (20)$$

where k_1, k_2 are constants. Give qualitative (but firm) arguments to show that the first term is due to the conduction electrons, and the second to lattice vibrations.

4. Consider an ideal Bose gas below the Bose-Einstein condensation temperature. Set the ground state energy to zero, and argue (starting from the Bose-Einstein distribution) that there will be a critical temperature T_c below which the number N_0 of atoms in the ground state is related to the total number of atoms N by

$$N_0 = N \left(1 - \left(\frac{T}{T_c} \right)^{\frac{3}{2}} \right) . \quad (21)$$

For the density of states use $g(\epsilon) = KV\sqrt{\epsilon}$, where K is some constant.

5. For the van der Waals gas close to the critical point, it is true that the difference in volume between gas and liquid vanishes according to

$$v_g - v_l \sim (t - 1)^{\frac{1}{2}} . \quad (22)$$

I am using dimensionless variables $t = T/T_{\text{cr}}$ etc, in terms of which the equation of state takes the form

$$\left(p + \frac{3}{v^2}\right)(3v - 1) = 8t . \quad (23)$$

Expanding this around $v = 1$ we get

$$p(v) = 4t - 3 - 6(t - 1)(v - 1) + 9(t - 1)(v - 1)^2 - \frac{3}{2}(9t - 8)(v - 1)^3 + \dots . \quad (24)$$

Using this, and Maxwell's equal area construction, derive the first relation!