

Tentamen i Analytisk Mekanik den 27 mars 2010, under tiden 9.00-15.00.
Lärare: Ingemar Bengtsson. Hjälpmedel: Penna, suddgummi och linjal.
Bedömning: 3 poäng/uppgift. Betyg: 0-3 = F, 4-6 = Fx, 7-9 = E, 10-12 = D, 13-14 = C, 15-16 = B, 17-18 = A.

1. Consider an action of the form

$$S[x(t)] = \int_{t_1}^{t_2} L(x, \dot{x}) dt, \quad (1)$$

where L is some function. Use the calculus of variations to derive the Euler-Lagrange equations.

2. Derive the inertia tensor of a homogeneous cube with respect to a corner. What are the moments of inertia?

3. The Hamiltonian is

$$H = \frac{1}{2m}(p_i - eA_i(x))(p_i - eA_i(x)). \quad (2)$$

Derive the Lagrangian by means of a Legendre transformation.

4. Check that, if the potential energy is $V = -\frac{k}{r}$, and L_i is the angular momentum vector, then the vector

$$M_i = \epsilon_{ijk} \dot{x}_j L_k - \frac{k}{r} x_i \quad (3)$$

is a constant of the motion. In what direction does it point? (Hint: Look at what happens at perihelion.)

5. Transform to a new coordinate system using $x^\mu \rightarrow x'^\mu = A^\mu{}_\nu x^\nu$. Transform all functions so that, in the new coordinate system, $f'(x') = f(x)$. Prove that

$$\frac{\partial f}{\partial x^\mu} \rightarrow \frac{\partial f'}{\partial x'^\mu} = \frac{\partial f}{\partial x^\nu} (A^{-1})^\nu{}_\mu. \quad (4)$$

6. In the two body problem one derives an effective potential for the radial motion, given by

$$E = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r) . \quad (5)$$

The Lagrangian for a heavy spinning top is

$$L = \frac{I_1}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\psi} + \dot{\phi} \cos \theta)^2 - Mgl \cos \theta . \quad (6)$$

Derive the form of the effective potential that governs the nutation (the motion in θ).

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1. Derrive the Lagrangian, given that the Hamiltonian is

$$H(q, p) = e^q \frac{p^2}{2} + \frac{q^2}{2} . \quad (7)$$

2. Find the maximum of the function

$$f(p_i) = - \sum_{i=1}^N p_i \ln p_i \quad (8)$$

under the conditions that $p_i \geq 0$ and

$$\sum_{i=1}^N p_i = 1 . \quad (9)$$

3. Prove that the time averages of the kinetic and potential energies of a particle bound in an $1/r$ potential obey

$$\langle 2T + V \rangle = 0 . \quad (10)$$

4. Show that any inertia tensor of a body confined to a plane can be reproduced by placing three equal masses at appropriate position in the plane.

5. Prove that the phase space of a Hamiltonian system has even dimension.

6. Transform all phase space functions according to

$$A(q, p) \rightarrow A'(q, p) = A(q, p) + \epsilon \{A(q, p), F\} + o(\epsilon^2) , \quad (11)$$

where $F = F(q, p)$ is an arbitrary phase space function. Check that this is a canonical transformation if you ignore all terms of order ϵ^2 or higher.

Tentamen i Analytisk Mekanik den tjugioandra augusti 2008, under tiden 9.00-15.00. Lärare: Ingemar Bengtsson. Hjälpmedel: Penna, suddgummi och linjal. Bedömning: 3 poäng/uppgift. Betyg: 0-3 = F, 4-6 = Fx, 6-9 = E, 10-12 = D, 13-14 = C, 15-16 = B, 17-18 = A.

1. Consider a particle not subject to forces, and change its velocity through the transformation $\delta x_i = v_i t$. Write down the Lagrangian and use Noether's theorem to deduce the corresponding constant of motion. Afterwards verify by explicit differentiation that it is indeed a (vectorial) constant of the motion, even though it depends explicitly on t .

2. Prove Kepler's third law, that the square of the period of a planet is proportional to its distance from the sun. The easy way to do this is to use the scaling properties of the Lagrangian

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - \frac{k}{r} . \quad (12)$$

3. In a rotating coordinate system

$$m\ddot{x}_i = F_i - 2m\epsilon_{ijk}\Omega_j\dot{x}_k + m(\Omega^2\delta_{ij} - \Omega_i\Omega_j)x_j . \quad (13)$$

Set up the equations for a Foucault pendulum (making many swings a day, so some terms can be ignored), and solve them.

4. State and prove a relation between the tensor of inertia relative to the center of mass, and relative to a point translated from the center of mass by the vector a_i .

5. Euler's equations for a freely spinning rigid top are

$$\begin{aligned} I_1\dot{\Omega}_1 + (I_3 - I_2)\Omega_2\Omega_3 &= 0 \\ I_2\dot{\Omega}_2 + (I_1 - I_3)\Omega_3\Omega_1 &= 0 \\ I_3\dot{\Omega}_3 + (I_2 - I_1)\Omega_1\Omega_2 &= 0 \quad , \end{aligned} \quad (14)$$

where Ω_i is the angular velocity around the i th principal axis. The Earth is flattened at the Poles, with ellipticity

$$\frac{I_1 - I_3}{I_1} \approx \frac{1}{300} . \quad (15)$$

Based on Euler's equations, what period do you expect for its (small) Chandler wobble?

6. Consider the Hamiltonian

$$H = \frac{1}{2m} p_i p_i \quad (16)$$

and the Poisson brackets

$$\{x_i, x_j\} = 0 \quad \{x_i, p_j\} = \delta_{ij} \quad \{p_i, p_j\} = e\epsilon_{ijk} B_k(x) . \quad (17)$$

Write down Hamilton's equations in "Poisson bracket form", and verify that they are the equations for a particle in an external magnetic field $B_i(x)$. Also check that these brackets really are Poisson brackets, in the sense that they are anti-symmetric and obey the Jacobi identity. To do the last part, you may find it helpful to write the magnetic field in terms of a vector potential,

$$\epsilon_{ijk} B_k = \partial_i A_j - \partial_j A_i . \quad (18)$$

Tentamen i Analytisk Mekanik den sjätte juni 2008, under tiden 9.00-15.00.
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1. Draw the phase space flow for the Hamiltonian

$$H = \frac{1}{2}p^2 + x(x-1)(x-2)(x-3) . \quad (19)$$

Your picture should show all fixed points clearly (but you do not have to locate them to two decimal places!). Into how many regions do the separatrices divide phase space?

2. Given an action $S = \int dt L(q, \dot{q})$. Suppose there exists a transformation $\delta q = \delta q(q, \dot{q})$ such that

$$\delta S = \int_{t_1}^{t_2} dt \frac{d\Lambda}{dt} \quad (20)$$

for some function $\Lambda = \Lambda(q, \dot{q})$. Prove that there exists a constant of the motion, and derive its form.

3. Compute the inertia tensor for a cube of constant mass density, with respect to a corner, and with respect to its center.

4. Euler's equations for a freely spinning top are

$$\begin{aligned} I_1 \dot{\Omega}_1 + (I_3 - I_2) \Omega_2 \Omega_3 &= 0 \\ I_2 \dot{\Omega}_2 + (I_1 - I_3) \Omega_3 \Omega_1 &= 0 \\ I_3 \dot{\Omega}_3 + (I_2 - I_1) \Omega_1 \Omega_2 &= 0 \end{aligned} . \quad (21)$$

where Ω_i is the angular velocity around the i th principal axis. Prove that the top can rotate around its principal axes, and find the conditions for these solutions to be stable. Solve the equations exactly for $I_2 = I_3$.

5. A Lagrangian for the central force two body problem is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r) . \quad (22)$$

Show how to reduce the equations of motion to two integrals that are doable (in principle) as soon as the function $V(r)$ is specified. Specialize to $V(r) = -kr^\alpha$, and deduce under what conditions on the exponent α circular orbits are stable.

6. A Lagrangian is

$$L = -\frac{1}{2}\left(1 - \frac{v}{r}\right)\dot{v}^2 + \dot{r}\dot{v} , \quad (23)$$

where v and r are configuration space coordinates. What are the canonical momenta? What is the Hamiltonian?