SL(2)

This might, perhaps, become lecture notes for a serious course, someday. I tried them in 2010. The illustrations have been taken from the classics.

At least apocryphically, Wigner once claimed that if it cannot be understood in terms of two by two matrices, it is not worth the trouble.[1] Presumably Wigner was thinking of group theory here. He certainly knew about it.

The set of all two by two matrices of determinant one,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} , \qquad ad - bc = 1 , \tag{1}$$

forms a *group*. The inverse of the given matrix can be written down by inspection,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . \tag{2}$$

It takes such a simple form because the determinant is assumed to equal unity.

Matrix groups consisting of matrices of determinant one are called special linear, but which particular group we obtain depends on the nature of the matrix elements:

a, b, c, d	Name of group
Complex	$SL(2, \mathbf{C})$
$d = \bar{a}, c = -\bar{b}$	$SU(2)$
$d = \bar{a}, c = \bar{b}$	$SU(1, 1)$
Real	$SL(2, \mathbf{R})$
Integers	$SL(2, \mathbf{Z})$
Integers modulo N	$SL(2, \mathbf{Z}_N)$

Here S is for special, L is for linear, and U is for unitary.

The *order* $|G|$ of a group G is the number of its elements. If the order is continuously ∞ and if in addition some technical conditions are obeyed (they are for all matrix groups) the group is a *Lie group*. Examples include $SL(2, \mathbf{C}), SL(2, \mathbf{R})$, $SU(1, 1)$, and $SU(2)$. The Lie group $SU(2)$ is known from quantum mechanics courses.

[1] If Wigner never said that, I can instead quote my teacher Arne Kihlberg.

Ex: Prove that $SU(1,1)$ is a group.

The order of $SL(2, \mathbf{Z})$ is countably ∞. This group is called the modular group. Clearly $SL(2, \mathbf{Z}) \in SL(2, \mathbf{R}) \in SL(2, \mathbf{C})$.

Ex: Prove that $SL(2, \mathbf{Z})$ is a group.

$\mathbf{Z}, \mathbf{R}, \mathbf{C}$ are *rings*—they consist of elements that can be added and multiplied, in such a way that they form a commutative (*abelian*) group under addition. $\mathbf{R}, \mathbf{C}$ are *fields* (Swedish *kropp*), meaning rings such that, if the identity element under addition is excluded, they also form abelian groups under multiplication. The integers were the original role model for rings, while the real (or rational) numbers were the original role model for the definition of fields. If you remember this it is easy to check if a given algebraic structure forms a ring, or a field.

Note that in the formula (2) for the inverse of a given matrix we need only the additive inverse of the entries. This is why we require that they belong to a ring, but not necessarily to a field.

Two integers are equal modulo N if they differ by a multiple of the integer N,

$$m = n \bmod N \quad \Leftrightarrow \quad m = n + kN \ . \tag{3}$$

Integers modulo N form a field if and only if $N = p$ is a prime number. This follows from a theorem in number theory: if m and N have largest common divisor d, then one can find integers r and s such that

$$rm + sN = d \ . \tag{4}$$

If $N = p$ then $d = 1$ for all non-zero m, and r is the multiplicative inverse of m modulo p. All finite fields are known (and there are some others). Note that this is not needed for $SL(2, \mathbf{Z}_N)$ to form a group.

Ex: Compute the order of $SL(2, \mathbf{Z}_p)$.

Matrices of determinant 1 preserve volume. This is often expressed by saying that they leave a certain tensor invariant. In this case it is the epsilon-tensor. The special linear group $SL(n)$ consists of N by N matrices such that

$$g^{b_1}{}_{a_1} g^{b_2}{}_{a_2} \dots g^{b_N}{}_{a_N} \epsilon_{b_1 b_2 \dots b_N} = \det g \epsilon_{a_1 a_2 \dots a_N} = \epsilon_{a_1 a_2 \dots a_N} \ . \tag{5}$$

Other groups are defined by the requirement that they leave some other tensor invariant. Of particular interest are tensors that define a non-degenerate quadratic form, that is expressions like

$$x^a g_{ab} y^a \ . \tag{6}$$

It is assumed that the matrix g_{ab} has an inverse g^{ab}, and it is convenient to bring it to a standard form (eg. $g_{ab} = \delta_{ab}$ if g_{ab} is symmetric).

Carrying on this theme we obtain a list of subgroups of the general linear group of invertible N by N matrices:

Invariant tensor	Property	Group
$\epsilon_{a_1 a_2 \dots a_N}$	anti-symmetric	special linear
ω_{ab}	anti-symmetric	symplectic
δ_{ab}	symmetric, pos. definite	orthogonal
η_{ab}	Minkowski metric	Lorentz

This defines the groups $Sp(N)$, $SO(N)$, and $SO(1, N-1)$ as subgroups of $SL(N)$. Note that the symplectic group $Sp(2) = SL(2, \mathbf{R})$. Isomorpisms between matrix groups typically happen only for low values of N.

Ex: Find the general form of matrices belonging to $SO(2) \in SL(2, \mathbf{R})$ and $SO(1, 1) \in SL(2, \mathbf{R})$.

Ex: How do you define $SU(2)$ and $SU(1, 1)$ from this point of view?

Why did we not consider the more general group $GL(2)$, defined such that $ad - bc \neq 0$? The answer is that this is not so interesting, since

$$g \in GL(2) \Rightarrow g = (\text{diagonal matrix}) \times (\text{matrix in } SL(2)) \ . \tag{7}$$

Because of this decomposition, once we understand SL we understand GL too. The diagonal matrices form the *center* of GL, i.e. they commute with everything.

The group acts on itself in three ways, left action $g \to g_1 g$, right action $g \to g g_1^{-1}$, and conjugation $g \to g_1 g g_1^{-1}$. Note that the inverse appears on the right because

$$g \to g g_1^{-1} \to g g_1^{-1} g_2^{-1} = g(g_2 g_1)^{-1} \ . \tag{8}$$

The action of the group on itself defines a *representation* of the group. In general a representation is a set of transformations of a set of objects which respects the multiplication table of the group. All matrix groups have a *defining representation* as matrices acting on some vector space, and all groups have a *trivial* (but not faithful) representation as the number 1.

If we choose a fixed group element g_1, then the set of all elements that can be written on the form $g g_1 g^{-1}$ for some $g \in G$ is called a *conjugacy class*. Any group can be partitioned into conjugacy classes, and the elements in a given conjugacy class have in a sense very similar properties. Compare Euler's theorem, according to which the rotation group can be partitioned into conjugacy classes such that each element in a given conjugacy class can be described as a rotation through some fixed angle about some unspecified fixed axis in space. The unit element is always a conjugacy class by itself.

We would like to divide $SL(2, \mathbf{R})$ into conjugacy classes. To do this it is helpful to know that any matrix M_1 can be brought into a standard *Jordan form* by means of conjugation, $M_1 \to M M_1 M^{-1}$. The catch is that Jordan's theorem assumes that an arbitrary complex matrix M can be used, while we want to restrict ourselves to real matrices. Anyway, given a matrix M_1, the first step is to solve the characteristic equation

$$\det(M_1 - \lambda) = 0 \ . \tag{9}$$

For n by n matrices this is an nth order polynomial, and it has n complex roots. If they are all different the matrix can be brought to diagonal form by means of conjugation (but the matrix M will typically not be a rotation matrix, or even a real matrix). If some eigenvalues are equal there may be entries equal to 1 just above the diagonal. Eg, if three eigenvalues are equal, the Jordan blocks

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} , \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} , \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} , \tag{10}$$

can appear, and cannot be transformed into each other. The information about the spectrum (the possible eigenvalues) can be summarised by the n independent quantities $\det M_1, \mathrm{Tr}M_1, \ldots, \mathrm{Tr}M_1^{n-1}$. Clearly their values cannot be changed by conjugation.

For 2 by 2 matrices the possible eigenvalues are determined by the determinant and the trace of the matrix. If the determinant equals one the eigenvalues must be $(\lambda_1, \lambda_2) = (re^{i\theta}, e^{-i\theta}/r)$. But for an $SL(2,\mathbf{R})$ matrix the trace must be real, so we are stuck with the two possibilites $(r, 1/r)$ or $(e^{i\theta}, e^{-i\theta})$—and evidently a real matrix with the second spectrum cannot be diagonalised using conjugation with matrices belonging to $SL(2,\mathbf{R})$. However, when the smoke clears, one finds the following representatives for the conjugacy classes of $SL(2,\mathbf{R})$:

$$-2 < \mathrm{Tr}g < 2 \ : \ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \quad 0 \leq \theta < 2\pi \tag{11}$$

$$2 < \mathrm{Tr}g \ : \ \begin{pmatrix} e^u & 0 \\ 0 & e^{-u} \end{pmatrix} , \quad 0 \leq u < \infty \tag{12}$$

$$\mathrm{Tr}g < -2 \ : \ -\begin{pmatrix} e^v & 0 \\ 0 & e^{-v} \end{pmatrix} , \quad 0 \leq v < \infty \tag{13}$$

$$\mathrm{Tr}g = 2 \ : \ \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} , \quad \sigma = -1, 0, 1 \tag{14}$$

$$\mathrm{Tr}g = -2 \ : \ \begin{pmatrix} -1 & \sigma \\ 0 & -1 \end{pmatrix} , \quad \sigma = -1, 0, 1 \tag{15}$$

So it is almost true that the conjugacy classes are labelled by the trace.

Ex: Work out the conjugacy classes of $SU(2)$ in the same way.

Ex: Use transformations in $SL(2,\mathbf{R})$ to diagonalise an arbitrary $SO(1,1)$ matrix.

Given a subgroup $H \in G$ one can divide the group G into *left cosets* of the form gH, where H is any element in the subgroup H and g is some fixed element of G. (Right cosets also exist.) Each group element belongs to exactly one left coset. If the group G has finite order it follows that the order of the subgroup must divide the order of the group (Lagrange's theorem). An *invariant* or *normal subgroup* is a subgroup $H \in G$ such that

$$gHg^{-1} = H \text{ for all } g \in G \tag{16}$$

(i.e. $ghg^{-1} \in H$ for all $h \in H$).

Ex: If H is an invariant subgroup, prove that the set of all left cosets gH forms a group in itself.

The resulting group is denoted G/H, and it is somehow a simpler group than the one we started out with. If the group G has finite order $|G|$, it follows that $|G/H| = |G|/|H|$.

The center of a group is always an invariant subgroup. A group that does not have any invariant subgroups at all is called a *simple group*. GL is not simple. Actually $SL(2)$ is not simple either, since it has $\pm\mathbf{1}$ as a non-trivial center. The groups

$$SL(2,\mathbf{C})/Z_2 \ , \quad SL(2,\mathbf{R})/Z_2 \ , \quad SU(2)/Z_2 \ , \tag{17}$$

etc, are simple groups. Here Z_2 denotes the cyclic group represented by the two elements ± 1. From quantum mechanics courses you know that $SU(2)/Z_2$ is really the same group as $SO(3)$. Frequently one sees the notation

$$PSL(2,\mathbf{C}) = SL(2,\mathbf{C})/Z_2 \ , \quad PSL(2,\mathbf{R}) = SL(2,\mathbf{R})/Z_2 \ . \tag{18}$$

The P stands for 'projective', meaning that a factor has been ignored somehow.

The projective groups are not only simple, they are often the physical relevant ones. This is so in quantum mechanics. The group that transforms between the physical states of a qubit is not $U(2)$, it is $PSU(2)$. The point is that, although one starts with a description of physical states using a two complex dimensional vector space on which the matrices act, it is decided that vectors differing by an overall phase represent the same physical states. So the group that acts on the states is $U(2)/I(2) = SU(2)/Z_2 = PSU(2)$, where $I(2)$ is the center of $U(2)$.

Literature: For rather more information than you need, consult I. N. Herstein, *Topics in Algebra*, 2nd ed. 1975.

The simple group $PSL(2, \mathbf{C})$ can be faithfully represented by the *Möbius transformations*

$$z \rightarrow z' = \frac{az + b}{cz + d} \ , \quad ad - bc \neq 0 \ . \tag{19}$$

This is a one-to-one mapping of the complex plane including the point at ∞ to itself.

Ex: Check that this is a representation of $SL(2, \mathbf{C})$.

To get the complete picture it is convenient to recall that the extended complex plane can be viewed as a sphere, with ∞ as an ordinary point at its South Pole. This construction, which is known as the sl Riemann sphere or (in arbitrary dimension) as the *stereographic projection*, works as follows: We start at the sphere end, and define the sphere by

$$X^2 + Y^2 + Z^2 = 1 \ . \tag{20}$$

We project from the South Pole, at $(X, Y, Z) = (0, 0, -1)$, to an Argand plane going through the equator and coordinatised by the complex number z. Geometrically one finds the unique straight line going through the South Pole and one other point of the sphere, and assigns to it the complex number defining the point where this line goes through the complex plane. In equations

$$z = \frac{X + iY}{Z + 1} \ , \tag{21}$$

or going the other way

$$X + iY = \frac{2z}{1 + |z|^2} \ , \quad Z = \frac{1 - |z|^2}{1 + |z|^2} \ . \tag{22}$$

This gives a one-to-one map between the complex plane and the sphere minus one point—the South Pole, which in fact corresponds to the ‘extra’ point ∞.

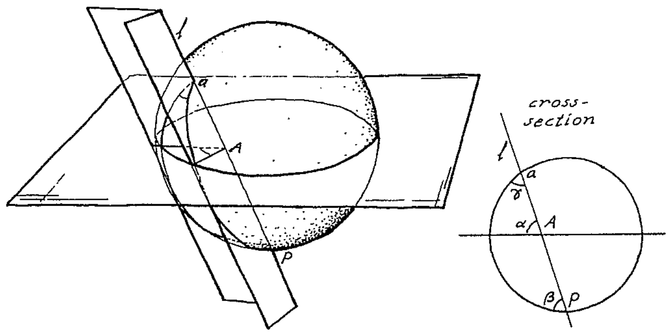


Figure 1: Why the stereographic projection is conformal. The picture was drawn by Sören Holst for his PhD thesis *Horizons and Time Machines*.

Ex: Let $z = \tan\frac{\theta}{2}e^{i\phi}$. What is (X, Y, Z)?

The stereographic projection is such that circles on the plane correspond to circles on the sphere, and straight lines on the plane correspond to circles passing through the South Pole on the sphere. Note that every straight line passes through ∞, so if ∞ is included straight lines are circles too. Angles between curves are preserved by the projection. In general maps that preserve angles are called *conformal*.

It is useful to have the Riemann sphere (or Bloch sphere, if we use the language of quantum information theory) in mind when thinking about what goes on in the extended complex plane.

Return to Möbius transformations. They have the following key properties:

1. They preserve angles between curves, and they are the most general one-to-one conformal mappings on the sphere.
2. They take circles to circles (where it is understood that a straight line counts as a circle too).
3. There is a unique Möbius transformation mapping any three points to any other set of three points.

I assume that this is at least vaguely familiar to you. Any transformation given by an analytic function is angle preserving, it is the one-to-one requirement that singles out the Möbius transformations.

Ex: Show explicitly that we can map any three points to $(0, 1, \infty)$.

We will be particularly interested in the case when a, b, c, d are real, and $ad - bc = 1$, that is when we have a representation of $SL(2, \mathbf{R})$. Obviously such Möbius transformations map the real line to itself. Moreover

$$i \rightarrow \frac{ai + b}{ci + d} = \frac{i(ad - bc) + ac + bd}{c^2 + d^2} = \frac{i + ac + bd}{c^2 + d^2} \ . \tag{23}$$

This has a positive imaginary part. By continuity it follows that the upper half plane is mapped into itself by $PSL(2, \mathbf{R})$.

Alternatively, let

$$z \rightarrow \frac{az + b}{\bar{b}z + \bar{a}} \ , \quad |a|^2 - |b|^2 = 1 \ . \tag{24}$$

We now have a representation of $SU(1, 1)$.

Ex: Check that these Möbius transformations take the unit circle into the unit circle, and the unit disk into the unit disk.

But we can find a Möbius transformation that takes the real line into the unit circle, namely

$$z \rightarrow \frac{z - i}{-iz + 1} \ . \tag{25}$$

This also takes the upper half plane into the unit disk. But if we first take the upper half plane into the unit disk, then apply a Möbius transformation taking the unit disk into itself, and finally take the unit disk back into the upper half plane, the result must be a Möbius transformation that takes the upper half plane into itself. In this way we establish a one-to-one correspondence between transformations in $PSU(1, 1)$ and transformations in $PSL(2, \mathbf{R})$. At the level of matrix groups a calculation shows that

$$\frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & i \\ i & 1 \end{array} \right) \left(\begin{array}{cc} a & b \\ \bar{b} & \bar{a} \end{array} \right) \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right) \in SL(2, \mathbf{R}) \ . \tag{26}$$

This establishes the isomorphism

$$SU(1, 1) \approx SL(2, \mathbf{R}) \ . \tag{27}$$

They are the same group in different guises.

Note that $SL(2,\mathbf{R})$ divides $\mathbf{C}\cup\infty$ into three *orbits*: the extended real line $\mathbf{R}\cup\infty$, the upper half plane, and the lower half plane, on each of which the group acts *transitively*, meaning that one can go from any point to any other point by means of a transformation in the group. The group $PSL(2,\mathbf{R})$ also acts *freely* on the orbits, meaning that for each group element there is a point that is moved by the transformation.

Let us now revisit the question of the conjugacy classes of $SL(2,\mathbf{R})$. The idea is that different conjugacy classes correspond to essentially different kinds of transformations. To understand a transformation it is natural to begin by asking for its fixed points. The fixed points of a Möbius transformation are determined by

$$z = \frac{az+b}{cz+d} \quad \Leftrightarrow \quad cz^2 + (d-a)z - b = 0 \ . \tag{28}$$

This is a quadratic equation and it will have two possibly coinciding roots. Hence every Möbius transformation has exactly two possibly coinciding fixed points, except for the identity $z' = z$ which leaves everything fixed. The case $c = 0$ may appear to be an exception too, but in fact it is not—if $c = 0$ one fixed point is ∞ and the other sits somewhere in the complex plane. If we assume that $c \neq 0$ we find the fixed points

$$z = \frac{1}{2c}\left(a - d \pm \sqrt{(a+d)^2 - 4}\right) \ . \tag{29}$$

Ex: Prove that a Möbius transformation is uniquely determined by its action on three points.

We now assume that a, b, c, d are real. Note that $a + d$ is the trace of the matrix. We then find three qualitively different cases:

1. $|\mathrm{Tr}g| > 2$: Two real fixed points.
2. $|\mathrm{Tr}g| < 2$: Two complex fixed points, one in each half plane.
3. $|\mathrm{Tr}g| = 2$: One real fixed point.

There are only three cases here because we are looking at $PSL(2,\mathbf{R}) = SL(2,\mathbf{R})/Z_2$, rather than at $SL(2,\mathbf{R})$ itself. The cases are called respectively

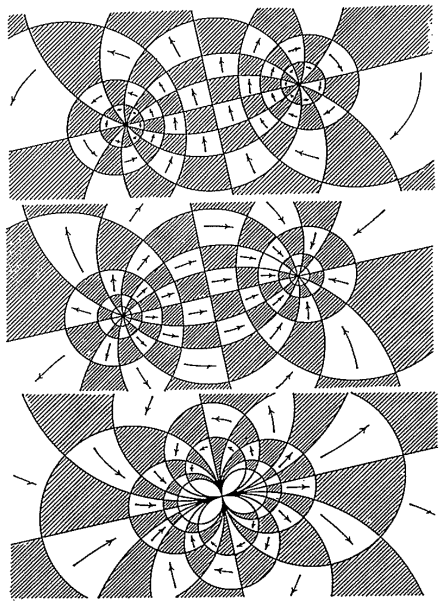


Figure 2: Elliptic, hyperbolic, and parabolic Möbius transformation. The picture illustrates Lester Ford's book on *Automorphic functions.*

hyperbolic, *elliptic*, and *parabolic* Möbius transformations. For $SL(2, \mathbf{C}$ we would also have to consider the case of complex trace.

Note that we could equally well study the conjugacy classes of $PSU(1,1)$, in which case the fixed points of a hyperbolic transformation lie on the unit circle, etc. We will pass freely between the pictures offered by these two groups.

To draw pictures of an elliptic Möbius transformation, place the fixed points at 0 and ∞ and use the $SU(1,1)$ picture. The transformation is $z' = e^{2i\phi}$, so the flow lines are those of a rotation in the unit disk. For the hyperbolic case, use the same fixed points and the $SL(2, \mathbf{R})$ picture. The transformation is $z' = e^{2u}z$, and you will see a set of straight flow lines diverging from the origin. The parabolic case is a bit trickier. If you place the fixed point at ∞ the transformation is $z' = z + b$. To see what goes on close to the fixed point, use a stereographic projection to the Riemann sphere. Having done this one can transform everything to illustrate the $SU(1,1)$ case, which is preferable since it is easier to get a feeling for the unit disk than for the upper half plane.

Ex: Prove that every hyperbolic Möbius transformation coming from $SU(1,1)$ contains exactly one flow line which is a circle or a straight line meeting the unit circle in right angles.

It is absolutely forbidden to do a calculation while doing this exercise.

This brings us to the subject of *non-Euclidean geometry*. Euclid formulated a number of axioms and postulates concerning points and straight lines. Paraphrasing slightly, the important ones say that through every pair of points there passes a unique straight line, and two lines meet in at most one point. The famous fifth postulate says that through any point not lying on a line there passes exactly one line which does not meet the given line. Such lines are called parallel. After many centuries of attempts to prove rather than assume the fifth postulate, it was suggested that one may formulate a consistent geometry in which all other axioms and postulates hold, but in which, given a line and a point not on this line, an infinite number of lines parallel with the given line pass through the given point.

Beltrami, and then Poincaré, offered a model which proves that non-Euclidean geometry is logically consistent, assuming only that Euclid's geometry itself is consistent. In this model space is the unit disk, its boundary not included. A point is a point in the unit disk. A line is a an arc of a circle, or a segment of a straight line, meeting the boundary of the disk in right angles. In this model it is easy to prove that two points determine a unique line, and that two lines meet in at most one point, but the parallel postulate is indeed modified in the manner suggested. The upper half plane can replace the unit disk in the model, if one prefers.

Ex: Prove that the sum of the angles of a triangle is less than $180°$.

The group $PSU(1,1)$ is very relevant here. Elliptic transformations work like rotations of the Poincaré disk, while hyperbolic transformations work like translations—to each line there corresponds a unique hyperbolic transformation taking the line to itself. Moreover one can find a pair of translations such that any point in the disk can be transformed into any other point. This is clearly very similar to what happens in Euclidean geometry. We can also try to define a notion of distance by insisting that the distance between any pair of points remains constant under all such transformations. It turns out

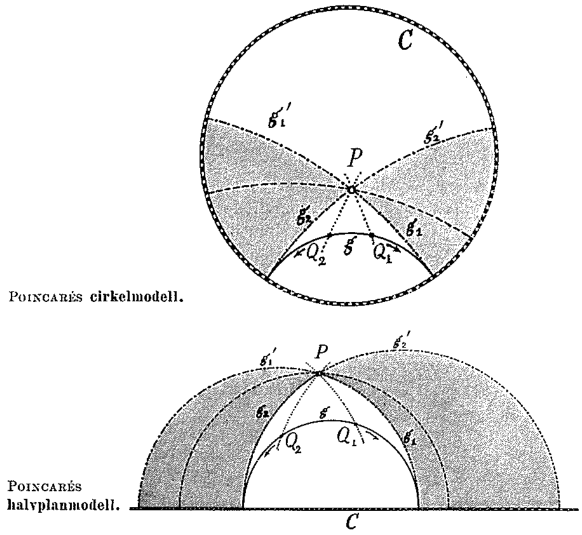


Figure 3: The modified fifth postulate in the two Poincaré models. The picture was drawn by K.-G. Hagstroem, to illustrate Marcel Riesz' book *En åskådlig bild av den icke-euklidiska geometrien.*

that there is one and only one way to do this. According to this definition the distance between a point on the unit circle and a point in the interior of the disk is always infinite. Also the sum of the angles of a triangle can be shown to depend on the area of the triangle in a simple way.

Literature: The standard reference is chapter one of L. R. Ford, *Automorphic functions*, 1929.

It is well known that $SU(2)/Z_2 \approx SO(3)$. This may lead one to suspect that $SU(1,1)$ is related to the Lorentz group $SO(1,2)$ in some way—at least they are similar in that both are three dimensional Lie groups. To explore this, introduce the symmetric matrix

$$\mathbf{x} = \begin{pmatrix} t+x & y \\ y & t-x \end{pmatrix} . \tag{30}$$

The parametrisation looks a bit odd, but we get a simple expression for the determinant,

$$\det x = t^2 - x^2 - y^2 . \tag{31}$$

This is the length squared of a vector in a three dimensional Minkowski space. Now choose a matrix $g \in SL(2, \mathbf{R})$, and compute

$$\mathbf{x} \to \mathbf{x}' = \begin{pmatrix} t'+x' & y' \\ y' & t'-x' \end{pmatrix} = g\mathbf{x}g^{\mathrm{T}} . \tag{32}$$

Note that we use the transpose on the right hand side. This is similar to our treatment of quadratic forms; the point here is that we want $\mathbf{x}'$ to be a symmetric matrix too. Clearly

$$t'^2 - x'^2 - y'^2 = \det g' = \det g = t^2 - x^2 - y^2 . \tag{33}$$

This is the defining property of a *Lorentz transformation*, so the vector (t, x, y) has been mapped to the vector (t', x', y') by a Lorentz transformation. Since the dimensions of the groups match, we have established the isomorphism

$$SL(2, \mathbf{R})/Z_2 = SO_0(1,2) . \tag{34}$$

The subscript is just fine print: We deal with the connected component of the Lorentz group, which means that we cannot change the sign of t.

Ex: Work out the explicit 3 by 3 matrix that acts on Minkowski space, as a function of the parameters in an $SL(2, \mathbf{R})$ matrix.

We can play the same game in four dimensions. Let $\mathbf{x}$ be a general Hermitean matrix

$$\mathbf{x} = \left(\begin{array}{cc} t+z & x-iy \\ x+iy & t-z \end{array} \right) . \tag{35}$$

Again the determinant is the length squared of a vector in Minkowski space, now of four dimensions. Pick a matrix $g \in SL(2, \mathbf{C})$, and compute

$$\mathbf{x} \rightarrow \mathbf{x}' = g\mathbf{x}g^\dagger . \tag{36}$$

This is again a Hermitean matrix, and again its determinant is preserved, so we have established the isomorphism

$$SL(2, \mathbf{C})/Z_2 = SO_0(1,3) . \tag{37}$$

The isomorphism $SU(2)/Z_2 = SO(3)$ is a special case of this, which arises if we restrict ourselves to transformations that preserve the trace of $\mathbf{x}$. The story ends here. There are no such isomorphisms for the higher dimensional Lorentz groups.

But why did the isomorphism happen in the first place? To understand it, we must understand how Minkowski space is divided into orbits under Lorentz transformations. If we pick a future pointing timelike vector of unit length as a *fiducial vector* on which we act with the Lorentz group, the resulting orbit consists of the hyperboloid

$$t^2 - x^2 - y^2 = 1 , \quad t > 0 . \tag{38}$$

This is a spacelike surface in Minkowski space. If the fiducial vector is spacelike the orbit is a timelike hyperboloid. If it is null (lightlike), the orbit is the forwards or backwards lightcone. The group acts freely on each of these orbits. Note that all points on a given orbit are at the same distance from the origin; in this sense the orbits behave like the orbits of the rotation group in ordinary space.

Rather than considering the lightcone as such, let us consider the set of all null rays through the origin. Again $SO_0(1,2)$ acts freely on this set. But the set of all null rays is clearly in one-to-one correspondence to a circle, which should ring a bell, since $PSU(1,1)$ also acts freely on a circle.

To describe the set of all null rays, note that a symmetric 2 by 2 matrix of determinant zero can be written as

$$x^{AB} = \pm u^A u^B \tag{39}$$

for a two dimensional vector $\mathbf{u}$.

Ex: Prove this. Prove a similar statement for an Hermitean matrix.

In this connection the two component vector is called a *spinor*. When $SL(2,\mathbf{R})$ acts on a matrix $\mathbf{x}$ of this form it acts separately on each spinor,

$$\mathbf{x} \to g\mathbf{x}g^{\mathrm{T}} \quad \Leftrightarrow \quad \mathbf{u} \to g\mathbf{u} \ . \tag{40}$$

Since we are interested in null rays rather than null vectors we identify matrices $\mathbf{x}$ that differ by an overall factor. Hence we will identify spinors differing by overall factors too. Momentarily closing our eyes to the fact that the second component may equal zero, we can label the set of all spinors up to a factor by a single real number x,

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \sim \frac{1}{x_1}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} \frac{x_0}{x_1} \\ 1 \end{pmatrix} \equiv \begin{pmatrix} x \\ 1 \end{pmatrix} \ . \tag{41}$$

Evidently

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \to \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \quad \Leftrightarrow \quad x \to \frac{ax+b}{cx+d} \ . \tag{42}$$

This is how real Möbius transformations act on null rays.

Ex: Show that spinors with the second component equal to zero correspond to the point ∞ on the real line.

We have divided the set of all non-zero spinors into equivalence classes, in such a way that all spinors that differ by a common factor are regarded as identical. This set is known as the *real projective line*, and can be identified with the set $\mathbf{R} \cup \infty$ on which real Möbius transformations act—or via an overall Möbius transformation to the unit circle. A rotation in Minkowski space corresponds to an elliptic Möbius transformation, and a boost to a hyperbolic Möbius transformation. The set of all null rays in 2+1 dimensions can be regarded as a real projective line.

We can treat complex spinors in the same way. We then end up with the *complex projective line*, which is identical to the Riemann sphere on which the group $SL(2,\mathbf{C})$ acts through Möbius transformations. The set of complex spinors up to overall factors can be identified with the set of null rays in a four dimensional Minkowski space, so this gives rise to the isomorphism $PSL(2,\mathbf{C}) \approx SO_0(1,3)$. In higher dimensions this trick fails: the set of all null rays is always a sphere, but the only spheres that are projective lines (over some number fields) are $\mathbf{S}^1$ and $\mathbf{S}^2$—and two more examples if you bring in something called quaternions and octonions, but we do not go into this here.

Return to 2+1 dimensions. There are other orbits in Minkowski space, notably the spacelike hyperboloids $t^2 - x^2 - z^2 = 1$. They are not so different from spheres in Euclidean space. In particular, any point on such a hyperboloid can be taken to any other point by means of a Lorentz transformation, just as any point on a sphere can be taken to any other by means of a rotation. Rotations preserve all distance relations in Euclidean space, and Lorentz transformations preserve all distance relations in Minkowski space. This means that, whatever impression one may get by drawing a picture of the hyperboloid, all its points are equivalent. There is no special point anywhere. Moreover the group of transformations that preserve distances on the hyperboloid is isomorphic to $SU(1,1)$, the group that—we claimed—preserves distances as defined in non-Euclidean geometry.

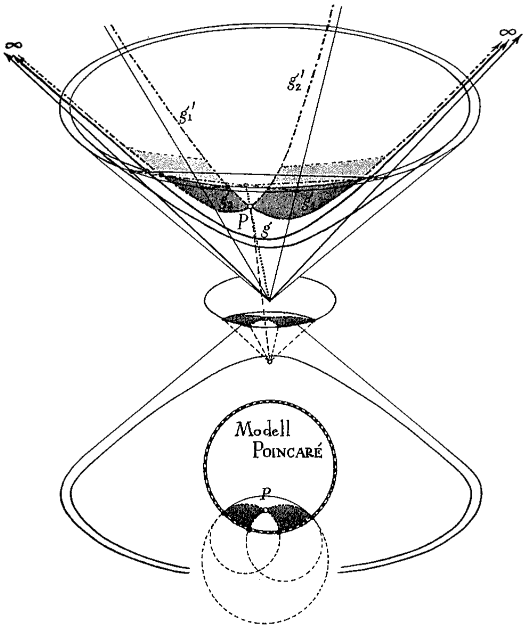
In effect we now have a second model of non-Euclidean geometry. The relation to the Poincaré disk model is given by a stereographic projection in Minkowski space. To set it up, consider the hyperboloid

$$T^2 - X^2 - Y^2 = 1 \ , \quad T > 0 \ , \tag{43}$$

in Minkowski space (we change notation slightly here), place a unit disk at $T = 0$ and centered at the origin, and project between the disk and the hyperboloid by means of straight lines from the point $(-1,0,0)$. This will give the formulæ

$$z = \frac{X + iY}{T + 1} \ , \tag{44}$$

or going the other way



Hyperboloidmodellen. Övriga modeller som projektioner av denna.

Figure 4: The hyperboloid and the Poincaré model. The picture is again by K.-G. Hagstroem, in Marcel Riesz' *En åskådlig bild av den icke-euklidiska geometrien.*

$$X + iY = \frac{2z}{1-|z|^2} \ , \quad T = \frac{1+|z|^2}{1-|z|^2} \ . \tag{45}$$

Note that we have to assume that $|z|^2 < 1$ for this to make sense.

We can now give an elegant interpretation of the 'lines' of the Poincaré model. Remember that a 'straight line' on a sphere, that is a *geodesic* on the sphere, is a Great Circle—the intersection of the sphere with a plane through the center of the sphere. Similarly a geodesic on the hyperboloid will be the intersection of the hyperboloid with a plane through the origin of Minkowski space, of the form

$$x_0T + x_1X + x_2Y = 0 \ , \tag{46}$$

where (x_0, x_1, x_3) must be a spacelike vector if the plane is going to intersect the hyperboloid at all.

Ex: Prove that a geodesic on the hyperboloid, so defined, corresponds to a line (a segment of a circle meeting the unit circle at right angles) on the Poincaré disk.

Clearly our definitions imply that geodesics on the hyperboloid transform among themselves under the Lorentz group. The same must therefore be true for the lines in the Poincaré model.

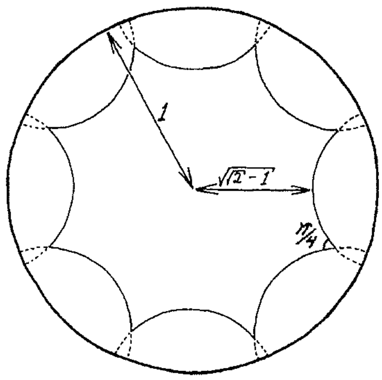


Figure 5: A regular octagon, of the size needed to create a smooth pretzel. The picture was drawn by Sören Holst for a paper in *Classical and Quantum Gravity.*

Using differential geometry one can show that the Riemann curvature scalar on the hyperboloid is constant and negative, while the Ricci tensor vanishes. Spaces of constant negative curvature plays an important role in mathematics. As an example, take a square in an Euclidean plane and identify opposite sides to form a flat torus; note that the vertices become a single point—of no special significance—on the torus. A similar trick is impossible in the Poincaré disk, because the sum of the angles of the square (or any quadrilateral) is less than 2π. Trying to identify opposite sides would lead to a torus with a non-smooth singular point. On the other hand, take an octagon in the flat plane. The sum of the angles at the vertices exceeds 2π, so we cannot identify opposite sides and obtain some smooth closed surface. However, in the Poincaré disk we can shrink the angles by letting the octagon grow, so there will be some critical size for which the operation of identifying opposite sides will succeed. The result turns out to have the topology of a pretzel, and a metric of constant negative curvature by construction.

Ex: Why did I not consider a hexagon?

The symmetry groups of the two planes—flat and negatively curved—actually played their roles here. The opposite sides of the square are identified by means of a translation taking one into the other. The same is true for the octagon: there is a unique (hyperbolic) Möbius transformation that is used to identify a pair of opposing edges—as a glance first at Fig. 2 and then at Fig. 5 should make clear.

Taking polygons with $4g$ edges in the Poincaré disk, and identifying the edges with suitable Möbius transformations, we obtain closed surfaces with constant negative curvature and $g > 1$ 'holes' in them. They are often referred to as *Riemann surfaces* of *genus* g. The torus and the sphere complete the list of Riemann surfaces.

Literature: A lovely perspective on all this is in R. Penrose: Relativistic symmetry groups, in A. O. Barut: *Group theory in non-linear problems*, 1974.

THE GROUP MANIFOLD OF $SL(2,\mathbf{R})$

The set of all elements of a continuous group form a space known as its *group manifold*. Let us see what this is for the group of all 2 by 2 matrices of determinant one. Any real 2 by 2 matrix can be parametrised as

$$g = \begin{pmatrix} U+Y & X+V \\ X-V & U-Y \end{pmatrix} . \qquad (47)$$

This will be an element of $SL(2,\mathbf{R})$ if and only if

$$\det g = -X^2 - Y^2 + U^2 + V^2 = 1 . \qquad (48)$$

The story is similar to that of $SU(2)$, but there is a significant difference too: the group manifold of $SU(2)$ is a *compact* space (in fact a 3-sphere), while that of $SL(2,\mathbf{R})$ is *non-compact*.

As we know, the group acts on itself according to

$$g \rightarrow g_{\mathrm{L}} g g_{\mathrm{R}}^{-1} . \qquad (49)$$

Since we may choose g_{L} and g_{R} independently this is a rather large group of transformations, needing $3 + 3 = 6$ parameters for its description. For a general Lie group G, it would be the group $G \times G$. Note we can use left action $g \rightarrow g_L g$ to go from any point in the group manifold to any other point; the group acts transitively on itself.

A fundamental idea is that one can use a suitably large set of transformations of a space to define its geometry, say because it may be the case that there is an essentially unique way of measuring distances that is preserved by the transformations. This idea works splendidly here. We begin by defining the *Maurer-Cartan form*

$$g^{-1}dg . \qquad (50)$$

This is a differential one-form defined on the group manifold, or if you like it defines a covariant vector field on the group.

Ex: Write this out explicitly using the coordinates X, Y, U, V.

The Maurer-Cartan form is invariant under left action by any fixed group element g_L;

$$g^{-1}dg \to (g_L g)^{-1}d(g_L g) = g^{-1}g_L^{-1}g_L dg = g^{-1}dg \ . \tag{51}$$

We can go on to define a group invariant metric

$$ds^2 = -\frac{1}{2}\text{Tr}\ g^{-1}dgg^{-1}dg \ . \tag{52}$$

Using the coordinates X, Y, U, V this is

$$ds^2 = -dX^2 - dY^2 + dU^2 + dV^2 \ . \tag{53}$$

To derive this we must remember that $d(-X^2 - Y^2 + U^2 + V^2) = 0$.

Ex: Prove that the group metric (52) is invariant also under right action, so that the isometry group of the group manifold G is $G \times G$.

From eq. (48) we already know that we can regard the group manifold of $SL(2, \mathbf{R})$ as a three dimensional hypersurface sitting inside a four dimensional space coordinatised by the *embedding coordinates* X, Y, U, V. We have now been led to a natural metric on this space, which in turn gives rise to a natural metrix on the group manifold. When viewed in this way the group manifold of $SL(2, \mathbf{R})$ is often referred to as 2+1 dimensional *anti-de Sitter space*.

Ex: Repeat all the above considerations for the group $SU(2)$.

In one special sense the story ends here—the only spheres that are group manifolds are the one and three dimensional spheres $\mathbf{S}^1$ and $\mathbf{S}^3$, and the only Lorentzian space that is a group manifold is 2+1 dimensional anti-de Sitter space. Although every Lie group has its group manifold, only very special spaces are group manifolds.

To understand anti-de Sitter space in a more intimate way, we introduce a set of three genuine coordinates (t, ρ, ϕ) through

$$(X, Y, U, V) = \left(\frac{2\rho\cos\phi}{1-\rho^2}, \frac{2\rho\sin\phi}{1-\rho^2}, \frac{1+\rho^2}{1-\rho^2}\cos t, \frac{1+\rho^2}{1-\rho^2}\sin t \right) . \tag{54}$$

The new coordinates range over

$$0 \leq t < 2\pi , \qquad 0 < \rho < 1 , \qquad 0 \leq \phi < 2\pi . \tag{55}$$

The coordinates (ρ, ϕ) are polar coordinates on a unit disk. The origin of the coordinate system coincides with the unit element of the group. The metric (52) now takes the form

$$ds^2 = \left(\frac{1+\rho^2}{1-\rho^2} \right)^2 dt^2 - dl^2 , \tag{56}$$

$$dl^2 = \frac{4}{(1-\rho^2)^2}(d\rho^2 + \rho^2 d\phi^2) . \tag{57}$$

To understand this, consider a surface at constant t. This is evidently an infinite spacelike surface, and the metric dl^2 on this surface is precisely the metric on the Poincaré disk. So far this is not all that different from Minkowski space—we have a stack, not of flat planes, but of planes with constant negative curvature. However, there is the peculiar feature that the time coordinate is periodic. Thus the topology of the group manifold is $\mathbf{S}^1 \times \mathbf{R}^2$ rather than the $\mathbf{R}^3$ topology enjoyed by Minkowski space. There is no grand father paradox, because there are no grandfathers on a group manifold.

The coordinates (t, ρ, ϕ) are called *sausage coordinates*, because they display anti-de Sitter space as a salami sliced by infinitely thin Poincaré disks.

Ex: Check that the spatial metric dl^2 really is the metric that one would obtain from the hyperboloid model of non-Euclidean geometry.

With this understanding we can revisit the question of conjugacy classes of $SL(2, \mathbf{R})$. Recall that a conjugacy class consists of all group elements that can be transformed into each other using $g \rightarrow g_1 g g_1^{-1}$ for some group element g_1. Since the same element occurs on the left and on the right, this is a proper subgroup of the full isometry group. We know that different conjugacy classes have different traces, and we also know that

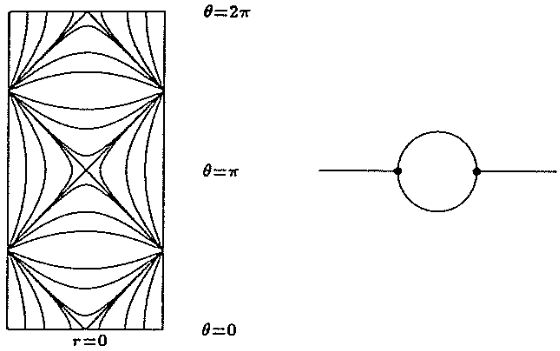


Figure 6: The conjugacy classes of $SL(2, \mathbf{R})$. The picture occurs in Joakim Hallin's PhD thesis *Some Aspects of Non-Perturbative Quantum Field Theory.* What we see on the left is a conformal diagram of anti-de Sitter space.

$$\mathrm{Tr}g = 2U = 2\frac{1+\rho^2}{1-\rho^2}\sin t \ . \tag{58}$$

Unless $U = \pm 1$ there is only one conjugacy class at the given value of U. But the equation that defines the group manifold can be rewritten

$$X^2 + Y^2 - V^2 = U^2 - 1 = -(1 - U^2) \ . \tag{59}$$

For $|U| < 1 \Leftrightarrow |\mathrm{Tr}g| < 2$ this is a spacelike hyperboloid. For $|U| > 1$ it is a timelike hyperboloid. For $|U| = 1$ it is a cone, and naturally splits into three pieces, consisting of the vertex, a forwards cone, and a backwards cone.

Ex: Redraw Fig. 6 for the group $PSL(2, \mathbf{R})$.

At constant V the conjugacy class has a metric given by

$$ds^2 = -dX^2 - dY^2 + dU^2 \quad \text{or} \quad dl^2 = dX^2 + dY^2 - dU^2 \ . \tag{60}$$

(I am willing to change the overall sign in order to get a positive definite rather than a negative definite metric on a spacelike surface.)

Let us now make more intimate use of the fact that $SL(2, \mathbf{R})$ is a Lie group. The unit element sits at $(X, Y, U, V) = (0, 0, 1, 0)$. Close to the unit element we find that

$$g \approx \begin{pmatrix} 1+x_2 & x_1+x_0 \\ x_1-x_0 & 1-x_2 \end{pmatrix} = \mathbf{1} + x_0\gamma_0 + x_1\gamma_1 + x_2\gamma_2 \ , \qquad (61)$$

where

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \qquad (62)$$

The determinant equals 1 to first order in the parameters x_i. The matrices γ_i form a basis for the set of all real traceless two by two matrices, and obey

$$[\gamma_1, \gamma_2] = -2\gamma_0 \ , \quad [\gamma_2, \gamma_0] = 2\gamma_1 \ , \quad [\gamma_0, \gamma_1] = 2\gamma_2 \ . \qquad (63)$$

This defines the *Lie algebra* of $SL(2, \mathbf{R})$: a vector space equipped with an anti-symmetric product, which can be thought of as the *tangent space* of the group at the origin.

Ex: Write down the commutator algebra enjoyed by the i times the Pauli matrices, $i\sigma_i$. This is the real Lie algebra of $SU(2)$. What is the key difference?

Let us now consider *one parameter subgroups* of $SL(2, \mathbf{R})$. An example is

$$g(\sigma) = e^{\sigma\gamma_0} = \mathbf{1} + \sigma\gamma_0 - \frac{\sigma^2}{2!}\mathbf{1} - \frac{\sigma^3}{3!}\gamma_0 + \ldots = \begin{pmatrix} \cos\sigma & \sin\sigma \\ -\sin\sigma & \cos\sigma \end{pmatrix} \ . \qquad (64)$$

The calculation is very simple because $\gamma_0^2 = -\mathbf{1}$. In the sausage coordinates this is a curve given by

$$(t, \rho, \phi) = (\sigma, 0, 0) \ . \qquad (65)$$

In fact this is a timelike *geodesic* through the origin (i.e. through the unit element of the group).

This did not happen by accident. For any Lie group we can write the one parameter subgroups

$$g(\sigma) = e^{\sigma\tilde{g}} \ , \qquad (66)$$

where $\tilde{g}$ is any element of the Lie algebra of the group. In the present case this must be a traceless two by two matrix, to ensure that the determinant of g equals unity. Since we assume that the group is a matrix group it is clear that $g(0)$ is its unit element, so this defines some curve through there. At the unit element this curve has the tangent vector

$$\lim_{\sigma\to 0}\frac{dg}{d\sigma}=\tilde{g}\ , \tag{67}$$

which shows that the Lie algebra indeed can be regarded as the vector space spanned by all tangent vectors at the origin. We know that, if we start from a given point in a direction determined by a given tangent vector, we will obtain a unique geodesic from these data. It is a theorem that, in a Lie group, the curves defined by the one parameter subgroups are indeed geodesics through the unit element in the group manifold.

If we want the geodesics through any other given point, we can use left action to move the geodesics we have already.

Ex: For $SU(2)$ physicists usually include, for good reasons, an imaginary factor i in the exponent of $g(\sigma)$. But since $SU(2)$ has a real Lie algebra, there is a reason for not doing it as well. What reason?

To get all the geodesics through the unit element, observe that

$$\begin{aligned} g(\sigma)=e^{x_0\gamma_0+x_1\gamma_1+x_2\gamma_2}= & \left(\begin{array}{cc}1 & 0\\ 0 & 1\end{array}\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots\right)+\\ & +\left(\begin{array}{cc}x_2 & x_1+x_0\\ x_1-x_0 & -x_2\end{array}\right)\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\dots\right)\ .\end{aligned} \tag{68}$$

For timelike geodesics, choose $(x_0,x_1,x_2)=\sigma(\hat{x}_0,\hat{x}_1,\hat{x}_2)$ so that

$$x^2\equiv x_0^2-x_1^2-x_2^2=\sigma^2(\hat{x}_0^2-\hat{x}_1^2-\hat{x}_2^2)=\sigma^2\ . \tag{69}$$

Thus $(\hat{x}_0,\hat{x}_1,\hat{x}_2)$ is a timelike unit vector in a Minkowski space. Then

$$g(\sigma)=\left(\begin{array}{cc}\cos\sigma+\hat{x}_2\sin\sigma & (\hat{x}_1+\hat{x}_0)\sin\sigma\\ (\hat{x}_1-\hat{x}_0)\sin\sigma & \cos\sigma-\hat{x}_2\sin\sigma\end{array}\right)\ . \tag{70}$$

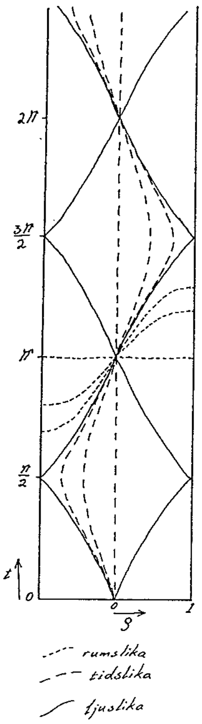


Figure 7: Geodesics on a two dimensional slice ($Y = 0$) of $SL(2, \mathbf{R})$, in sausage coordinates. The picture was drawn by Sören Holst for his Master's Thesis *Om existensen av slutna tidslika kurvor konstruerade med hjälp av massor i 2+1 dimensioner.*

Comparing to eq. (47) we see that these curves can be written as

$$(X(\sigma), Y(\sigma), U(\sigma), V(\sigma)) = (\hat{x}_1 \sin\sigma, \hat{x}_2 \sin\sigma, \cos\sigma, \hat{x}_0 \sin\sigma) \ . \tag{71}$$

The rather more complicated expression for the geodesics in sausage coordinates can be worked out from here. One interesting fact is that the entire family of timelike geodesics will refocus at the point $(X, Y, U, V) = (0, 0, -1, 0)$.

Spacelike geodesics are obtained similarly; unlike the timelike ones they are not periodic in σ. To get the null geodesics, set

$$(x_0, x_1, x_2) = \sigma(1, \cos\phi, \sin\phi) \quad \Rightarrow \quad x^2 = 0 \ . \tag{72}$$

Then

$$g(\sigma) = \begin{pmatrix} 1 + \sigma \sin \phi & \sigma(\cos \phi + 1) \\ \sigma(\cos \phi - 1) & 1 - \sigma \sin \phi \end{pmatrix} . \tag{73}$$

Hence

$$(X(\sigma), Y(\sigma), U(\sigma), V(\sigma)) = (\sigma \cos \phi, \sigma \sin \phi, 1, \sigma) , \tag{74}$$

confirming that $U = 1$ is the light cone from the unit element.

With a full understanding of the topology, geometry, isometry group, and geodesics of the group manifold, we are satisfied. Moreover all that we did generalises straightforwardly to any matrix Lie group. The only thing that changes—usually dramatically—is the length of the calculations.

Literature: For the geometry of 2+1 dimensional anti-de Sitter space, see S. Holst, *Horizons and Time Machines*, 2000.

REPRESENTATION THEORY

Given a group, we are interested in representing it using matrices with complex entries in every possible way. To cut this down a little, we are interested in *inequivalent irreducible* representations only. By definition a representation is reducible if the vector space on which the matrices act contains a proper subspace that is left invariant by all matrices in the representation. It can be shown that for all finite groups (and for all compact Lie groups) every representation admits a choice of basis in the vector space such that the matrices take a block diagonal form, where each block defines an irreducible representation. Moreover each *irrep*—as they are are commonly abbreviated—can be assumed to be unitary. That is

- Every representation of a finite or compact group is equivalent to a direct sum of irreducible ones.
- There is a scalar product such that a given irrep is unitary.

Two representations are said to be equivalent if the matrices in one representation can be turned into those in the other by means of a basis change in the space on which they act. The first statement implies that we can bring all the matrices of a given matrix into the form

$$U(g) = \left(\begin{array}{c|c|c|c} U^{(1)}(g) & \mathbf{0} & \dots & \mathbf{0} \\ \hline \mathbf{0} & U^{(2)}(g) & \dots & \mathbf{0} \\ \hline \vdots & \vdots & & \vdots \\ \hline \mathbf{0} & \mathbf{0} & \dots & U^{(k)}(g) \end{array} \right) . \tag{75}$$

The second statement is clearly plausible. For any group element g in a finite group there exists an integer n such that $g^n = \mathbf{1}$, hence the eigenvalues of a matrix representing g must be roots of unity, as they would be if the matrix were unitary. To prove it, let $\langle v|w\rangle_0$ be any scalar product. Then we define a new scalar product by means of a sum over all group elements,

$$\langle v, w\rangle = \frac{1}{|G|} \sum_{g\in G} \langle U(g)v, U(g)w\rangle_0 \ . \tag{76}$$

Ex: Prove that the sum does indeed define a scalar product.

Ex: Prove that $U(g)$ is unitary with respect to the new scalar product.

Fortunately we do not have to perform this sum, it is enough to know that we lose no generality by restricting ourselves to unitary representations. We can generalise to any compact Lie group—which by definition has a group manifold of finite volume—if we replace the sum with an integral. This disposes of $SU(2)$, but at this point we still know nothing about non-compact groups such as $SL(2, \mathbf{R})$.

Note that a representation does not have to be *faithful*. Indeed the trivial representation, in which all group elements are represented by the number 1, is available for any group.

The task of finding all inequivalent irreducible representations can still be a hard problem, but for finite groups there are some general theorems to guide us. It can be shown that:

- The number of inequivalent irreducible representations of G equals the number of conjugacy classes of G.
- Let the dimensions of the distinct irreducible representations be n_i. Then

$$\sum_i n_i^2 = |G| \ . \qquad (77)$$

- The dimensions n_i divide $|G|$.
- A simple group has only one one-dimensional irrep.

I should now introduce characters, and then focus on the finite group $SL(2, \mathbf{Z}_p)$, with p a prime number. Another point to be made is that the representation theory of the non-compact group $SO(1, 2)$ is more interesting than that of the compact group $SU(2)$.

THE MODULAR GROUP

In which I should show how $SL(2, \mathbf{Z})$ can be used to tesselate the upper half plane, introduce modular forms, discuss snow crystals, and more.