Closed String Spectrum :

In this case one deals with two copies of oscillators,

$$\boxed{[\alpha^\mu_m, \alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}, \; [\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \quad [x^\mu, p^\nu] = i\eta^{\mu\nu}}$$

and for each copy, the prodedure mirrors the open string case. We define the number operators,

$$\boxed{N_R = \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_m \;, \qquad N_L = \sum_{m=1}^{\infty} \bar{\alpha}_{-m}\cdot\bar{\alpha}_m}$$

with respect to these, α^μ_n for $n \geq 1$ are lowering operators and α^μ_{-n} are raising operators. States are labeled by their N_R and N_L eigenvalues and their p^μ eigenvalue. The ground state $|0,0\rangle$ or simply $|0\rangle$ is defined by:

$$\boxed{\alpha^\mu_m|0\rangle = 0, \quad \bar{\alpha}^\mu_m|0\rangle = 0 \quad \text{for all } m \geq 1}$$

The state can also carry a momentum k^μ:

$$p^\mu|0,k\rangle = k^\mu|0,k\rangle$$

Each $\alpha^\mu_{-n_1}$ and $\bar{\alpha}^\mu_{-n_2}$ acting on $|0,k\rangle$ increases the N_R and N_L eigenvalues by n_1 and n_2 respectively. The basis states are then of the form

$$\boxed{|\{i_{(\mu,m)}\}, \{\bar{i}_{(\mu,m)}\}, k\rangle = \prod_{m\geq 1}^{\infty}\prod_{\mu=0}^{D-1} (\alpha^\mu_{-m})^{i_{(\mu,m)}} (\alpha^\mu_{-m})^{\bar{i}_{(\mu,m)}} |0,k\rangle}$$

and are specified by the sets of integers $\{i_{(\mu,m)}\}$ and $\{\bar{i}_{(\mu,m)}\}$ for each μ and m.

$$N_R \,|\{i_{(\mu,m)}\},\{\bar{i}_{(\mu,m)}\},k\rangle = \sum_{\mu,m} m\, i_{(\mu,m)}\, |\{i_{(\mu,m)}\},\{\bar{i}_{(\mu,m)}\},k\rangle$$

$$N_L \,|\{i_{(\mu,m)}\},\{\bar{i}_{(\mu,m)}\},k\rangle = \sum_{\mu,m} m\, \bar{i}_{(\mu,m)}\, |\{i_{(\mu,m)}\},\{\bar{i}_{(\mu,m)}\},k\rangle$$

A large number of these states have negative norm (as seen in the open string case, these are states associated with α^0_{-m} and $\bar{\alpha}^0_{-m}$). Also at this stage, there is no connection between these quantum states of a free, massless field theory in 2-dimensions and the quantum dynamics of the closed string. Both these issues are resolved by the imposition of the Virasoro constraints on this space of states.

The Virasoro generators are given by

$$\boxed{L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{m-n}\cdot\alpha_n:, \qquad \bar{L}_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\bar{\alpha}_{m-n}\cdot\bar{\alpha}_n:}$$

For $m \neq 0$, the normal ordering is redundant. But for $m = 0$, the normal ordered L_0 & $\bar{L}_0$ differe from the expressions without normal ordering by an undetermined constant a. Hence in all quantum expressions we replace L_0 & $\bar{L}_0$ by $(L_0 - a)$ and $(\bar{L}_0 - a)$ to take this ambiguity into account. Noting that for closed strings,

$$\alpha_0^\mu = p^\mu / 2\sqrt{\pi T} ,$$

one has,

$$\boxed{L_0 = \frac{1}{8\pi T} p^\mu p_\mu + N_R , \qquad \bar{L}_0 = \frac{1}{8\pi T} p^\mu p_\mu + N_L}$$

After a somewhat lengthy computation, one can show that the L_m's & $\bar{L}_m$'s satisfy the Virasoro algebra,

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12}(m^3-m)\,\delta_{m+n,0}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n) \bar{L}_{m+n} + \frac{D}{12}(m^3-m)\,\delta_{m+n,0}$$

The computation will **not be** reproduce**d** here but can be found in the standard literature.

A state $|\psi\rangle$ is regarded as a physical state if it satisfies

$$(L_m - a\delta_{m,0})|\psi\rangle = 0 \; ; \qquad (\bar{L}_m - a\delta_{m,0})|\psi\rangle = 0$$

$$\text{for } m \geq 0$$

These guarantee that for a physical state, $\langle\psi| L_m - a\delta_{m,0}|\psi\rangle = 0$ and $\langle\psi|\bar{L}_m - a\delta_{m,0}|\psi\rangle = 0$ for all m. These are the quantum mechanical generalizations of $(T_{\alpha\beta})_{\text{classical}} = 0$. It turns out that states satisfying these conditions do not have negative norm only for <u>$a = 1$, $D = 26$</u>. Furthermore, these constraints establish a connection between the physics of the string and the state space.

First consider the $m=0$ constraints: these can also be rewritten as,

$$\boxed{(L_0+\bar{L}_0-2a)|phys\rangle=0, \qquad (L_0-\bar{L}_0)|phys\rangle=0}$$

leading to the following operator equations,

$$\boxed{M^2\equiv -p^\mu p_\mu=(4\pi T)\,(N_R+N_L-2a)\,, \qquad N_R=N_L}$$

or, using $\alpha'=1/2\pi T$ and $a=1$,

$$\boxed{\alpha' M^2=2(N_R+N_L-2)\,, \qquad N_R=N_L}$$

From here one can read off the mass spectrum of physical states. Besides this, physical states must also satisfy

$$L_m\,|phys\rangle=0, \quad \bar{L}_m|phys\rangle=0\,, \quad m\geq 1\,.$$

A few examples are:

$N_L=N_R=0$: This state is $|0,k\rangle$ with $-\alpha' k^2=\alpha' M^2=-4$, and thus corresponds to a tachyon in closed string theory. Its prensence indicates that closed bosonic string theory, at least the way it is formulated now, is not stable.

$N_L=N_R=1$: These states are of the form

$$\varepsilon_{\mu\nu}(k)\,\alpha^\mu_{-1}\,\bar{\alpha}^\nu_{-1}|0,k\rangle$$

with $M^2=-k^2=0$. The L_1 and $\bar{L}_1$ conditions imply that $$\varepsilon_{\mu\nu} k^{\mu} = \varepsilon_{\mu\nu} k^{\nu} = 0$$

These constraints insure that physical states always have non-negative norms $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu}$. The discussion very closely parallels the open string case and will not be repeated here. Suffice it to say that the negative contributions to the norm $\varepsilon_{\mu\nu}\varepsilon^{\mu\nu}$, i.e., $-\varepsilon_{0i}\varepsilon_{0i} - \varepsilon_{i0}\varepsilon_{i0}$ cancel against positive contributions from $\varepsilon_{00}\varepsilon_{00}$ and a part of $\varepsilon_{ij}\varepsilon_{ij}$, say, $\varepsilon_{11}\varepsilon_{11}$. Thus, effectively only the 24 oscillators α^i_{-1} and $\bar{\alpha}^i_{-1}$ for $(i = 2, 3, \ldots, 26)$ contribute to the physical states which thus have 24×24 physical components.

$\varepsilon_{\mu\nu}(k)$ is a rank 2 tensor of $SO(1,25)$ forming a reducible representation of the group. The irreducible representations to which it decomposes are the symmetric traceless tensor $\varepsilon_{(\mu\nu)}(k)$, the antisymmetric tensor $\varepsilon_{[\mu\nu]}(k)$ and the trace $\varepsilon^{\mu}{}_{\mu}$ which is a scalar. The states associated with $\varepsilon_{(\mu\nu)}$ are the gravitons as suggested by their transformation properties. In terms of the Fourier transform $\varepsilon_{(\mu\nu)}(x)$, the constraints are

$$\partial_\mu \partial^\mu \varepsilon_{(\rho\sigma)}(x) = 0, \qquad \partial^\mu \varepsilon_{(\mu\nu)} = \partial^\nu \varepsilon_{(\mu\nu)} = 0$$

which correspond to the linearized Einstein equation for the metric $g_{\rho\sigma} = \eta_{\rho\sigma} + \varepsilon_{(\rho\sigma)}(x)$ in the covariant gauge (That this identification is valid beyond the linearized level will be discussed below).

$\varepsilon_{[\mu\nu]}(x)$ is referred to as the Kalb-Ramond field and the trace part is related to a field called the dilaton. We will see more of these below.

Vertex Operators and Space-Time Physics

So far we have considered the theory of a single string and shown that it can only exist in certain quantum states characterized by mass, spin (i.e., representation under the Lorentz group), etc. We have not yet described how these strings interact with each other. We will now give an elementary discussion of this issue and then, use the machinery to build a picture of space-time physics in terms of the physics of the string worldsheet.

The basic object in terms of which string interactions are formulated is the "vertex operator": In a string interaction, a given physical state $|\psi\rangle$ (that could correspond to a number of strings in their physical oscillation modes, $|\psi\rangle = |\text{phys}\rangle_{\text{String 1}} \otimes |\text{phys}\rangle_{\text{String 2}} \otimes \cdots$)

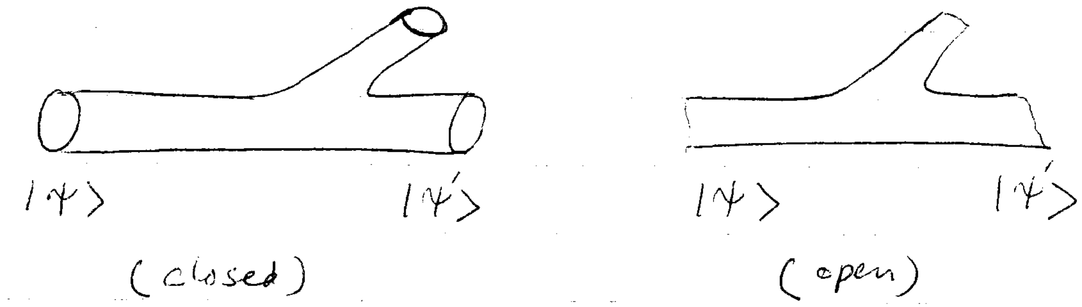
evolves into another physical state $|\psi'\rangle$. The operator that affects the change from $|\psi\rangle$ to $|\psi'\rangle$ is called a vertex operator,

$$|\psi'\rangle = V|\psi\rangle$$

V is constructed in terms of functions $X^\mu(\sigma,\tau)$ on the string worldsheet and its form is constrained by the requirement that both $|\psi\rangle$ and $|\psi'\rangle$ should satisfy the physical state conditions,

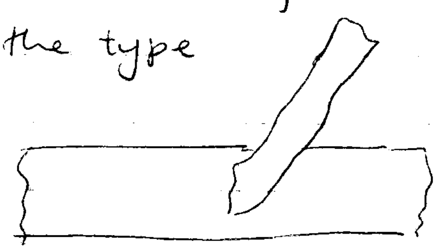
$$(L_m - a\,\delta_{m,0})\,|\text{phys}\rangle = 0$$

Diagramatically, the interaction corresponds to the process of joining and splitting of closed or open strings:



Using conformal transformations (which are symmetries of the theory), the above worldsheets can be mapped to

Under this map, the region where the worldsheet branches out is mapped to a point (denoted by a cross x) on the original worldsheet. It is at this point that the vertex operator corresponding to the emission of the new state is inserted (we will see later what this actually means in concrete terms). Note that for open strings, this insertion takes place only on the boundary of the worldsheet, i.e., open string vertex operators are constructed in terms of $X^\mu(\sigma=0,\tau)$ or $X^\mu(\sigma=\ell/2,\tau)$. A process of the type



(not allowed)

is not allowed since at the splitting point, the worldsheet is not a two dimensional manifold. Let us now study the construction of vertex operators in more detail.

Conformal Weights and Conformal Primaries

After gauge fixing reparametrizations and Weyl transformations, we are left with conformal transformations

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+), \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-)$$

as the symmetry group of the worldsheet theory. We are interested in fields that have nice conformal transformation properties. A field $\phi(\sigma^+, \sigma^-)$ that transforms as

$$\phi(\sigma^+, \sigma^-) = \left(\frac{\partial \tilde{\sigma}^+}{\partial \sigma^+}\right)^{\bar{h}} \left(\frac{\partial \tilde{\sigma}^-}{\partial \sigma^-}\right)^{h} \tilde{\phi}(\tilde{\sigma}^+, \tilde{\sigma}^-)$$

is called a conformal primary field of conformal weight $(h, \bar{h})$. For example, a covariant 2d tensor with p "+" and q "−" indices $\phi_{++\cdots+--\cdots-}$ will have conformal weights (q, p) since,

$$\phi_{\underbrace{++\cdots+}_{p}\underbrace{--\cdots-}_{q}}(\sigma^+, \sigma^-)(d\sigma^+)^p(d\sigma^-)^q = \tilde{\phi}_{\underbrace{++\cdots+}_{p}\underbrace{--\cdots-}_{q}}(\tilde{\sigma}^+, \tilde{\sigma}^-)(d\tilde{\sigma}^+)^p(d\tilde{\sigma}^-)^q$$

Since the only dynamical variable we have at our disposal is the field $X^\mu(\sigma^+,\sigma^-)$, conformal primaries are built in terms of X^μ and its derivatives. Since X^μ itself is a scalar on the worldsheet, under a conformal transformation,

$$X'^\mu(\sigma^+,\sigma^-) = \tilde{X}^\mu(\tilde{\sigma}^+,\tilde{\sigma}^-) \qquad \text{(classically)}$$

Hence it has $(h,\bar{h}) = (0,0)$ if treated as a classical field. Here are some fields and their <u>classical conformal dimensions</u>:

	X^μ	$\partial_+ X^\mu$	$\partial_- X^\mu$	$e^{ik_\mu X^\mu}$	$\partial_+ X^\mu \partial_+ X_\mu$	$\partial_- X^\mu \partial_- X_\mu$	$\partial_+^2 X^\mu$
h	0	0	1	0	0	2	0
$\bar{h}$	0	1	0	0	2	0	2

We will see below that in the quantum theory $(h,\bar{h})$ may differ from their classical values. Before this, one should explain how a conformal transformation is implemented in the quantum theory. For this, let us consider an infinitesimal conformal transformation

$$\tilde{\sigma}^+ = \sigma^+ + \varepsilon^+(\sigma^+), \qquad \tilde{\sigma}^- = \sigma^- + \varepsilon^-(\sigma^-)$$

Then to first order in $\varepsilon^\pm$,

$$\phi(\sigma^+,\sigma^-) = \left(1 + \bar{h}\,\partial_+\varepsilon^+ + \cdots\right)\left(1 + h\,\partial_-\varepsilon^- + \cdots\right)\Big(\tilde{\phi}(\sigma^+,\sigma^-) + \partial_+\phi\,\varepsilon^+ + \partial_-\phi\,\varepsilon^- + \cdots\Big)$$

or

$$\boxed{\delta\phi = \tilde{\phi} - \phi = -(\bar{h}\,\partial_+ \varepsilon^+ + \varepsilon^+ \partial_+)\phi - (h\,\partial_- \varepsilon^- + \varepsilon^- \partial_-)\phi} \qquad (1)$$

consider the case of closed and open strings separately:

Closed strings: In this case $\varepsilon^+(\sigma^+)$ and $\varepsilon^-(\sigma^-)$ are independent and so are the corresponding variations:

$$\delta_{\varepsilon^-}\phi = -(h\,\partial_- \varepsilon^- + \varepsilon^- \partial_-)\phi$$

$$\delta_{\varepsilon^+}\phi = -(\bar{h}\,\partial_+ \varepsilon^+ + \varepsilon^+ \partial_+)\phi \qquad (2)$$

Remember that the invariance of the theory under conformal transformations resulted in the existance of two conserved charges Q_{ε^-} and Q_{ε^+} (earlier we denoted these by Q and $\bar{Q}$). The Noether's theorem also tells us that these conserved charges act as generators of the symmetry transformation,

$$\tilde{\phi} = e^{-iQ_\varepsilon}\,\phi\, e^{iQ_\varepsilon}$$

$$= (1 - iQ_\varepsilon + \cdots)\,\phi\,(1 + iQ_\varepsilon + \cdots) = \phi - i[Q_\varepsilon, \phi] + \cdots$$

where we use the fact that ε is infinitesimal and Q_ε is linear in ε. Here ε could stand for ε^+ or ε^-. Hence quantum mechanically, the variations $\delta_{\varepsilon^-}\phi$ and $\delta_{\varepsilon^+}\phi$ are computed from

$$\delta_{\varepsilon^-}\phi = -i\left[Q_{\varepsilon^-}, \phi\right]$$

$$\delta_{\varepsilon^+}\phi = -i\left[Q_{\varepsilon^+}, \phi\right]$$

comparing this with the classically derived expressions, we obtain

$$\boxed{\begin{aligned} i\left[Q_{\varepsilon^-}, \phi\right] &= \left(h\,\partial_-\varepsilon^- + \varepsilon^-\partial_-\right)\phi \\ i\left[Q_{\varepsilon^+}, \phi\right] &= \left(\bar{h}\,\partial_+\varepsilon^+ + \varepsilon^+\partial_+\right)\phi \end{aligned}} \qquad (3)$$

These equations allow us to compute the conformal weights of ϕ by evaluating the commutators on the left hand side.

Remember that for closed strings,

$$Q_{\varepsilon^+} = T\int_0^l d\sigma\, T_{++}\,\varepsilon^+$$

$$T_{++} = \frac{2\pi}{l^2 T}\sum_k \bar{L}_k\, e^{-2\pi i k(\tau+\sigma)/l}$$

$$\varepsilon^+ = \sum_{k'} \varepsilon^+_{k'}\, e^{2\pi i k'(\tau+\sigma)/l}$$

with similar expressions for Q_{ε^-}. Then,

$$\boxed{\begin{aligned} Q_{\varepsilon^+} &= \frac{2\pi}{l}\sum_k \varepsilon^+_k\,\bar{L}_k \\ Q_{\varepsilon^-} &= \frac{2\pi}{l}\sum_k \varepsilon^-_k\,L_k \end{aligned}} \qquad (4)$$

For the modes $\varepsilon_k^{\pm}$ of the transformation, equations (3) reduce to

$$\boxed{\begin{aligned} [L_k , \phi] &= e^{2\pi i k \sigma^-/l} \left(kh - \frac{il}{2\pi} \partial_-\right) \phi \\ [\bar{L}_k , \phi] &= e^{2\pi i k \sigma^+/l} \left(k\bar{h} - \frac{il}{2\pi} \partial_+\right) \phi \end{aligned}} \qquad (5)$$

$(h, \bar{h})$ can be read off from these equations, although in practice computing the commutators on the left hand side could be quite tedious (later we will recast equations (3) in terms of the operator product expansion which will considerably simplify the calculations).

Let us expand $\phi(\sigma^+, \sigma^-)$ as a function of either σ^+ or σ^-

$$\begin{aligned} \phi(\sigma^+, \sigma^-) &= \sum_m \phi_m(\sigma^+)\, e^{-2\pi i m (\tau - \sigma)/l} \\ &= \sum_m \bar{\phi}_m(\sigma^-)\, e^{-2\pi i m (\tau + \sigma)/l} \end{aligned}$$

(we need only one of these expansions at a time). Substituting in (5) and equating the coefficients of, say, $e^{-2\pi i n \sigma^{\pm}/l}$ on both sides of the equations, we get

$$\boxed{\begin{aligned} [L_k , \phi_n] &= (k(h-1) - n)\, \phi_{k+n} \\ [\bar{L}_k , \bar{\phi}_n] &= (k(\bar{h}-1) - n)\, \bar{\phi}_{k+n} \end{aligned}} \qquad (6)$$

These can be used to read off the conformal dimensions of fields.

As an application of these equations, let us consider a field $\phi^{(2)}$ of conformal weights $h=2$, $\bar{h}=0$. Then we should have

$$[L_k, \phi_n^{(2)}] = (k-n)\, \phi_{k+n}^{(2)}$$

where $\phi_n^{(2)}$ are the Fourier modes of $\phi^{(2)}$. Now consider T_{--} (with L_n as Fourier modes) which at the classical level has $h=2$, $\bar{h}=0$. The analogue of the above equation is now the Virasoro algebra,

$$[L_k, L_n] = (k-n)\, L_{k+n} + \frac{D}{12}(k^3-k)\,\delta_{k+m,0}$$

which differs from the equation for $\phi_n^{(2)}$ due to the presence of the central charge term. Thus the non-zero central charge indicates that T_{--} is not a good conformal primary field at the quantum level. Another example of this kind is X^μ which has $\bar{h}=0, h=0$ in the classical theory, but is not a conformal primary in the quantum theory.

Open Strings : Now we obtain the formulae that compute the conformal dimensions of fields in open string theory. We have seen that under an infinitesimal conformal transformation

$$\delta\sigma^+ = \varepsilon^+(\sigma^+), \qquad \delta\sigma^- = \varepsilon^-(\sigma^-),$$

a primary field ϕ changes by

$$\delta_\varepsilon\phi = -\left(\bar{h}\,\partial_+\varepsilon^+ + h\,\partial_-\varepsilon^- + \varepsilon^+\partial_+ + \varepsilon^-\partial_-\right)\phi$$

In open string theory ε^+ and ε^- are not independent and are given in terms of the same Fourier modes ε_k :

$$\varepsilon^+ = \sum_k \varepsilon_k\, e^{2\pi i k(\tau+\sigma)/l}, \qquad \varepsilon^- = \sum_k \varepsilon_k\, e^{2\pi i k(\tau-\sigma)/l}$$

On the other hand, $\delta_\varepsilon\phi$ is also given by

$$\delta_\varepsilon\phi = -i\left[Q_\varepsilon\,,\,\phi\right]$$

where Q_ε is the generator of conformal transformations in open string theory,

$$Q_\varepsilon = T\int_0^{l/2} d\sigma\left(T_{++}\,\varepsilon^+ + T_{--}\,\varepsilon^-\right) = T\int_0^{l} d\sigma\; T_{++}\,\varepsilon^+$$

$$= \frac{2\pi}{l}\sum_k \varepsilon_k L_k$$

Then,

$$\sum_k \varepsilon_k\left[L_k\,,\,\phi\right] = \frac{l}{2\pi i}\left(\bar{h}\,\partial_+\varepsilon^+ + h\,\partial_-\varepsilon^- + \varepsilon^+\partial_+ + \varepsilon^-\partial_-\right)\phi$$

We are interested in this equation at the boundary of the open string worldsheet i.e. for $\sigma=0$ or $\sigma=\ell/2$ (since we will regard ϕ as a vertex operator which as argued above should live on the worldsheet boundary). Then, using

$$\partial_\pm \varepsilon^\pm = \sum_k \frac{2\pi i k}{\ell} \varepsilon_k e^{2\pi i k(\tau\pm\sigma)/\ell}$$

and comparing the coefficients of ε_k on both sides, we get

$$\boxed{[L_k, \phi]_{\sigma=0} = e^{2\pi i k \tau/\ell}\left(J k - \frac{i\ell}{2\pi}\partial_\tau\right)\phi(\tau,\sigma=0)}$$

where $J = h+\bar{h}$ is the conformal dimension of ϕ in open string theory. Using the expansion

$$\phi(\tau, \sigma=0) = \sum \phi_n e^{-2\pi i n\tau/\ell}$$

$$\sum_n [L_k, \phi_n]\, e^{-2\pi i n\tau/\ell} = \sum_{n'} e^{2\pi i(k-n')\tau/\ell}(Jk-n')\,\phi_{n'}$$

$\Rightarrow$

$$\boxed{[L_k, \phi_n] = ((J-1)k-n)\,\phi_{k+n}} \qquad (7)$$

This is the equation that can be used to determine the conformal dimension J of fields in open string theory.