Stress-energy tensor & the gauge fixed action

So far, we have defined $T_{\alpha\beta}$ in terms of the variation of S_P with respect to $h^{\alpha\beta}$. The constraint $T_{\alpha\beta}=0$ was then necessary to insure that the free, massless scalar fields X^μ contained the dynamics of the string at the classical level.

After gauge fixing $h_{\alpha\beta}$ to $\eta_{\alpha\beta}$, one can no longer vary S_P with respect to the metric, although the constraint $T_{\alpha\beta}\Big|_{h=\eta}=0$ still has to be imposed.

Does $T_{\alpha\beta}$ have any intrinsic significance in the flat 2-dim. theory obtained after gauge fixing (without making a reference to the metric)? The answer is contained in the Noether's theorem. For latter reference, we describe this in a more general set up, giving the proof afterwards.

Noether's Theorem (statement):

Consider an n-dimensional space-time with coordinates y^α, $\alpha=0,1,\cdots,n-1$ and a collection of fields $\phi^i(y)$, where the extra indices have been suppressed. The field theory action has the form

$$S=\int_\Omega d^n y\, \mathcal{L}\big(\phi^i(y),\, \partial_\alpha\phi^i(y)\big)$$

where $\mathcal{L}$ is the Lagrangian density and the action has been evaluated over a finite region of space-time Ω. Consider infinitesimal transformations of coordinates and fields,

$$\boxed{\begin{aligned} & y^\alpha \longrightarrow y'^\alpha = y^\alpha + \delta y^\alpha \\ & \phi^i(y) \longrightarrow \phi'^i(y'), \quad \phi'^i(y) - \phi^i(y) = \bar{\delta}\phi^i \end{aligned}}$$

The change in ϕ^i could be partly due to δy^α and partly due some intrinsic transformation of the fields. Note that $\bar{\delta}\phi^i$ is calculated with both ϕ'^i and ϕ^i having the same argument y^α. Noether's theorem states that under such transformations, the variation of the action is given by

$$\delta S = \int_\Omega d^n y \left(\partial_\alpha J^\alpha\right)$$

with

$$\boxed{J^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^i)} \bar{\delta}\phi^i + \mathcal{L}\,\delta y^\alpha}$$

The invariance of the action ($\delta S = 0$) then implies
→ the current conservation equation, (for any $\bar{\delta}$ & δy)

$$\boxed{\partial_\alpha J^\alpha = 0}$$

This in turn allows us to define a conserved charge,

$$\boxed{Q = \int_\Sigma d^{n-1}y \; J^0 , \qquad \frac{\partial Q}{\partial y^0} = 0}$$

The integration region Σ is the space part of the space-time volume Ω (more precisely, a spatial section). The <u>time independence of Q</u> follows from

$$\frac{\partial Q}{\partial y^0} = \int_\Sigma d^{n-1}y \; \partial_0 J^0 = -\int_\Sigma d^{n-1}y \sum_{\alpha=1}^{n-1} \partial_\alpha J^\alpha \equiv -\int_\Sigma d^{n-1}y \; \vec{\nabla}\cdot\vec{J}$$

$$= -\oint_{\partial\Sigma} d\vec{S}\cdot\vec{J} = 0$$

Here, $\vec{J}$ contains the spatial components of J^α and we have used the <u>divergence theorem</u> to write the volume integral over Σ as the surface integral over the <u>boundary</u> of Σ, denoted by $\partial\Sigma$. Then the last step follows provided $\vec{J}$ vanishes on the spatial boundary $\partial\Sigma$, which is easy to arrange if the boundary lies at infinity (as in most 3+1 dimensional field theories). However, it may also happen that $\partial\Sigma$ is not at infinity and $\vec{J}$ does not vanish on the boundary. In that case the content of the theory should be arranged such that the integral $\oint_{\partial\Sigma} d\vec{S}\cdot\vec{J}$ <u>vanishes nevertheless</u>. A third possibility is that the space Σ does not have a boundary ($\partial\Sigma = 0$), for example $\Sigma = S^1$ (a circle). In that case the integral over $\partial\Sigma$ vanishes trivially.

Remark : The variations δy^α and $\bar\delta\phi^i$, in general, depend on a number of arbitrary parameter that can be collectively denoted by $\{\epsilon\}$. The way we have defined the conserved currents and charges, they will also depend on $\{\epsilon\}$, hence it is more logical to denote them by J^α_ϵ and Q_ϵ. Since $\{\epsilon\}$ are arbitrary, we do not want physical quantities to depend on them. When the transformations are given explicitly in terms of $\{\epsilon\}$, they will appear as multiplicative factors in J^α_ϵ and Q_ϵ. Hence, we can simply remove them by hand and define physical currents and charges, independent of $\{\epsilon\}$.

Noethers Theorem (Proof) :

Under the transformations $y^\alpha \to y'^\alpha = y^\alpha + \delta y^\alpha$ and $\phi^i(y) \to \phi'^i(y')$, the variation of the action is given by

$$\delta S = \int_{\Omega'} d^n y' \, \mathcal{L}\left(\partial'_\alpha \phi'(y'), \phi'(y')\right) - \int_{\Omega} d^n y \, \mathcal{L}\left(\partial_\alpha \phi(y), \phi(y)\right)$$

$$= \int_{\Omega'} d^n y \, \mathcal{L}\left(\partial_\alpha \phi'(y), \phi'(y)\right) - \int_{\Omega} d^n y \, \mathcal{L}\left(\partial_\alpha \phi(y), \phi(y)\right)$$

where, we use the fact that y' is a dummy integration variable. Now, using

$$\phi'^i(y) = \phi^i(y) + \bar\delta\phi^i \, , \quad \text{one has}$$

$$\mathcal{L}\left(\partial_\alpha \phi'(y), \phi'(y)\right) = \mathcal{L}\left(\partial_\alpha \phi + \partial_\alpha(\bar{\delta}\phi), \phi + \bar{\delta}\phi\right)$$

$$\approx \mathcal{L}(\partial_\alpha \phi, \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^i)} \partial_\alpha(\bar{\delta}\phi^i) + \frac{\partial \mathcal{L}}{\partial \phi^i} \bar{\delta}\phi^i$$

$$\approx \mathcal{L}(\partial_\alpha \phi, \phi) + \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^i} \bar{\delta}\phi^i\right) - \underbrace{\left[\partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial \partial_\alpha \phi^i}\right) - \frac{\partial \mathcal{L}}{\partial \phi^i}\right]}_{=0 \text{ (by Eqn. of motion)}} \bar{\delta}\phi^i$$

so that,

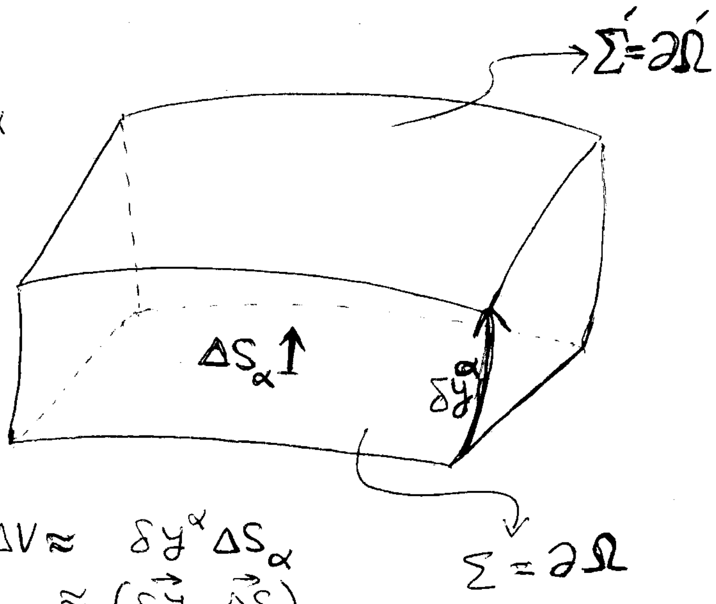
$$\delta S = \int_{\Omega'} d^n y\, \mathcal{L} - \int_{\Omega} d^n y\, \mathcal{L} + \int_{\Omega'} d^n y\, \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^i)} \bar{\delta}\phi^i\right)$$

$$\approx \int_{\Omega' - \Omega} d^n y\, \mathcal{L} + \int_{\Omega} d^n y\, \partial_\alpha\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^i)} \bar{\delta}\phi^i\right) + O(\delta y\, \bar{\delta}\phi)$$

$\Omega' - \Omega$ is the volume bounded between the boundaries of Ω' and Ω. Since the boundary of Ω is shifted to the boundary of Ω' by the coordinate shift δy^α, we have,

$$\Omega' - \Omega = \int_{\Sigma} dS_\alpha\, \delta y^\alpha$$

where $\{dS_\alpha\}$ denotes an area element on the boundary Σ of Ω



Thus, comparing with

$$\Omega' - \Omega \equiv \int_{\Omega'-\Omega} d^n y \approx \int_{\Sigma} dS_\alpha \, \delta y^\alpha$$

we have,

$$\int_{\Omega'-\Omega} d^n y \, \mathcal{L} = \int_{\Sigma} dS_\alpha \left(\delta y^\alpha \mathcal{L}\right) = \int_{\Omega} d^n y \, \partial_\alpha \left(\mathcal{L} \, \delta y^\alpha\right)$$

where the last step again follows from the divergence theorem. Hence,

$$\delta S = \int_{\Omega} d^n y \; \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^i)} \bar{\delta}\phi^i + \mathcal{L} \, \delta y^\alpha \right)$$

$$= \int_{\Omega} d^n y \left(\partial_\alpha J^\alpha \right)$$

from which the Noether theorem stated earlier follows.

2-d Space-time transformations in the worldsheet theory

We now consider the gauge fixed Polyakov action S_P. In this case,

$$n = 2, \quad y^\alpha \to \sigma^\alpha, \quad \phi^i \to X^\mu.$$

Assume that the action is invariant under

$$\boxed{\sigma^\alpha \to \sigma'^\alpha = \sigma^\alpha + \delta\sigma^\alpha}$$

Since X^μ are 2-d scalars,

$$X'^\mu(\sigma'^\alpha) = X^\mu(\sigma^\alpha)$$

$\Rightarrow$

$$X'^\mu(\sigma^\alpha) + \partial_\alpha X^\mu \delta\sigma^\alpha = X^\mu(\sigma^\alpha) \text{ or } \boxed{\bar{\delta} X^\mu = -\partial_\beta X^\mu \delta\sigma^\beta}$$

Then,

$$J^\alpha = -\frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} \partial_\beta X^\mu \delta\sigma^\beta + \mathcal{L}\delta\sigma^\alpha$$

or

$$J_\alpha = \left(-\eta_{\alpha\lambda}\frac{\partial \mathcal{L}}{\partial(\partial_\lambda X^\mu)} \partial_\beta X^\mu + \eta_{\alpha\beta}\mathcal{L}\right)\delta\sigma^\beta$$

Using $\quad S_P = \int d^2\sigma \mathcal{L}_P, \quad \mathcal{L}_P = -\frac{T}{2}\eta^{\tilde{\alpha}\tilde{\beta}}\partial_{\tilde{\alpha}} X^{\tilde{\mu}} \partial_{\tilde{\beta}} X^{\tilde{\nu}} \eta_{\tilde{\mu}\tilde{\nu}}$

one can verify that,

$$\boxed{J_\alpha = T(T_{\alpha\beta}\delta\sigma^\beta)}$$

(T = string tension)

where

$$\boxed{\begin{aligned} T_{\alpha\beta} &= \frac{-1}{T}\left(\frac{\partial \mathcal{L}_P}{\partial(\partial^\alpha X^\mu)} \partial_\beta X^\mu - \eta_{\alpha\beta}\mathcal{L}_P\right) \\ &= \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\left(\eta^{\tilde{\alpha}\tilde{\beta}}\partial_{\tilde{\alpha}} X^\mu \partial_{\tilde{\beta}} X_\mu\right)\end{aligned}}$$

which coincides with the earlier expression. Thus in flat space-time, $T_{\alpha\beta}$ is related to the conserved currents corresponding to space-time symmetries $\delta\sigma^\alpha$, $\delta S_P = 0$

The current conservation $\partial_\alpha J^\alpha = 0$ allows us to define a conserved charge

$$\rightarrow \qquad Q = \overset{?}{(-)} \int_0^l J^0 \, d\sigma = \overset{?}{(+)} -T \int_0^l (T_{0\beta} \, \delta\sigma^\beta) \, d\sigma$$

Remember that σ is the "space" coordinate on the worldsheet and we ~~consider~~ work in a parametrization where it has a finite range $0 \leqslant \sigma \leqslant l$.

Now

$$\boxed{\frac{\partial Q}{\partial \tau} = 0 \qquad \text{provided} \qquad J^1\Big|_{\sigma = l} - J^1\Big|_{\sigma = 0} = 0}$$

Thus a conserved charge Q exists only when $J^1(\sigma,\tau)$ **satisfies** the appropriate **boundary** conditions. Since J^1 is a function of $X^\mu(\sigma,\tau)$, this implies appropriate boundary conditions for X^μ. We will latter discuss such boundary conditions from a somewhat different point of view.

Let us now consider some special cases of coordinate transformations:

Translations: $\qquad \sigma^\alpha \rightarrow \sigma^\alpha + c^\alpha$ (constant c's)

Then, $J_\alpha = T_{\alpha\beta} c^\beta$ and $\partial^\alpha J_\alpha = 0 \Rightarrow \partial^\alpha T_{\alpha\beta} = 0$.

The conserved quantities are:

for $\tau \rightarrow \tau + c^0$; $\qquad H = -\int_0^l J^0 d\sigma = +T\int_0^l T_{00} \, d\sigma = +T\int_0^l d\sigma \, (T_{++} + T_{--})$

(sign ?) $\qquad \sigma \rightarrow \sigma + c^1$: $\qquad \overset{(?)}{\rightarrow} P = \int_0^l J^0 d\sigma = -T\int_0^l T_{01} \, d\sigma = -T\int_0^l d\sigma \, (T_{++} - T_{--})$

(sign of H is **chosen** to make it the Hamiltonian)

Rotations (Lorentz transformations):

$$\delta\sigma^{\alpha} = \omega^{\alpha}{}_{\lambda}\,\sigma^{\lambda}\,, \qquad (\omega^{\alpha\lambda} = -\omega^{\lambda\alpha})$$

$$\partial^{\alpha} J_{\alpha} = \partial^{\alpha}\left(T_{\alpha\beta}\,\omega^{\beta}{}_{\lambda}\,\sigma^{\lambda}\right) = T_{\alpha\beta}\,\omega^{\beta}{}_{\lambda}\,\eta^{\alpha\lambda} = T_{\alpha\beta}\,\omega^{\beta\alpha} = 0$$

$$\Rightarrow \qquad \boxed{T_{\alpha\beta} = T_{\beta\alpha}}$$

Scale transformations:

$$\delta\sigma^{\alpha} = \lambda\,\sigma^{\alpha}\,, \qquad \left(\sigma^{\alpha} \to (1+\lambda)\,\sigma^{\alpha}\right)$$

$$J_{\alpha} = \lambda\,T_{\alpha\beta}\,\sigma^{\beta}\,, \qquad \partial^{\alpha}J_{\alpha} = 0 \Rightarrow \qquad T_{\alpha\beta}\,\eta^{\alpha\beta} = T^{\alpha}{}_{\alpha} = 0$$

Summary: translation, rotation and scaling invariances imply

$$\boxed{\partial^{\alpha}T_{\alpha\beta} = 0, \qquad T_{\alpha\beta} = T_{\beta\alpha}\,, \qquad T^{\alpha}{}_{\alpha} = 0}$$

World sheet hamiltonian: $H = T\int_0^{\ell} d\sigma\,(T_{++} + T_{--})$

World sheet momentum: $P = T\int_0^{\ell} d\sigma\,(T_{--} - T_{++})$

H and P are conserved provided the associated current component J^1 satisfies the appropriate boundary conditions.

Stress-energy tensor and Conformal invariance

The gauge fixed S_P is invariant under conformal transformations

$$\delta\sigma^+ = \varepsilon^+(\sigma^+), \quad \delta\sigma^- = \varepsilon^-(\sigma^-)$$

The associated conserved current is

$$J_\alpha = T\, T_{\alpha\beta}\, \delta\sigma^\beta \Rightarrow \begin{cases} J_+ = T\, T_{++}\, \varepsilon^+ \\ J_- = T\, T_{--}\, \varepsilon^- \end{cases}$$

with

$$\partial^\alpha J_\alpha = 0 \qquad \text{or} \qquad \partial_+ J_- + \partial_- J_+ = 0$$

One can define charges,

$$\boxed{\begin{aligned} Q &= \int_0^l d\sigma\, J_0 = \int_0^l d\sigma\,(J_+ + J_-) \\ &= T\int_0^l d\sigma\,(T_{++}\,\varepsilon^+ + T_{--}\,\varepsilon^-) \end{aligned}} \qquad (1)$$

These charges are conserved ($\frac{\partial Q}{\partial \tau} = 0$), provided,

$$\boxed{J'\Big|_{\sigma=0}^{\sigma=l} \equiv (J_+ - J_-)\Big|_{\sigma=0}^{\sigma=l} \equiv (T_{++}\,\varepsilon^+ - T_{--}\,\varepsilon^-)\Big|_{\sigma=0}^{\sigma=l} = 0} \qquad (2)$$

Since $\varepsilon^\pm$ are arbitrary functions, in effect, we have infinite number of conserved charges. Latter we will look at their structure in more detail.

Under certain conditions, one can define an even larger class of conserved charges: Note that $J_+ = T\, T_{++}\, \varepsilon^+$ is a function only of σ^+ and $J_- = T\, T_{--}\, \varepsilon^-$ depends only on σ^-. Hence,

$$\partial_+ J_- = \partial_\tau (J_-) + \partial_\sigma (J_-) = 0$$

$$\partial_- J_+ = \partial_\tau (J_+) + \partial_\sigma (-J_+) = 0$$

If we define currents

$$j^\alpha_{(+)} = \{ J_+ , -J_+ \}$$

$$j^\alpha_{(-)} = \{ J_- , J_- \} \qquad \text{for } \alpha = 0,1$$

then we have two conserved currents $j^\alpha_{(+)}$ and $j^\alpha_{(-)}$ instead of one (J^α):

$$\partial_+ J_- = \partial_\alpha j^\alpha_{(-)} = 0 \qquad (\text{for } \alpha = 0,1)$$

$$\partial_- J_+ = \partial_\alpha j^\alpha_{(+)} = 0$$

The corresponding charges are

$$\boxed{\begin{aligned} Q_{(-)} &= \int_0^l d\sigma\, j^0_{(-)} = T \int_0^l d\sigma\, T_{--}\, \varepsilon^- \\ Q_{(+)} &= \int_0^l d\sigma\, j^0_{(+)} = T \int_0^l d\sigma\, T_{++}\, \varepsilon^+ \end{aligned}} \qquad (3)$$

These charges are conserved, $\frac{\partial Q_{(\pm)}}{\partial \tau} = 0$, provided

$$j^1_{(\pm)}\Big|_{\sigma=0}^{\sigma=l} = 0, \quad \text{or},$$

$$\boxed{\begin{aligned} \frac{1}{T}(J_+)\Big|_{\sigma=0}^{\sigma=l} &= T_{++}\,\varepsilon^+\Big|_{\sigma=0}^{\sigma=l} = 0 \\ \frac{1}{T}(J_-)\Big|_{\sigma=0}^{\sigma=l} &= T_{--}\,\varepsilon^-\Big|_{\sigma=0}^{\sigma=l} = 0 \end{aligned}} \qquad (4)$$

Thus, if boundary conditions (4) can be satisfied, then we have two classes of conserved charges, $Q_{(+)}$ and $Q_{(-)}$ (3). The charge $Q = Q_{(+)} + Q_{(-)}$ of (1) then becomes redundant. We will see latter that this corresponds to having <u>closed strings</u> so that all fields are <u>periodic in σ: $X^\mu(\sigma) = X^\mu(\sigma+l)$.</u>

If it is not possible to satisfy (4), then conserved $Q_{(+)}$ and $Q_{(-)}$ charges do not exist. However, it may still be possible to satisfy the boundary condition (2) and define conserved charges (1). This corresponds to the case of <u>open strings</u>. We will soon encounter the issue of boundary conditions in a different guise and return to this problem again.

Mode Expansion of $T_{\pm\pm}$

Let us now look at charges $Q_{(\pm)}$ in more detail:

$$Q_{(\pm)} = T \int_0^l d\sigma \; T_{\pm\pm} \, \varepsilon^{\pm}$$

Each choice of the functions $\varepsilon^{\pm}$ leads to a different set of conserved charges $Q_{(\pm)}$, hence there are infinite conserved quantities. These can be easily parametrized as follows:

On the interval $0 \leq \sigma \leq l$, one has the complete and orthonormal set of basis functions $\exp(2\pi i n \sigma/l)$ for $n = -\infty, \ldots, +\infty$. The σ-dependence of $\varepsilon^{\pm}$ can then be expanded in a Fourièr sêries. However, since these depend on σ^+ and σ^-, the τ-dependence is fixed by the σ-dependence. Hence,

$$\boxed{\begin{aligned} \varepsilon^+(\sigma^+) &= \sum_{n=-\infty}^{\infty} e^{-2\pi i n(\tau+\sigma)/l} \, \varepsilon_n^+ \\ \varepsilon^-(\sigma^-) &= \sum_{n=-\infty}^{\infty} e^{-2\pi i n(\tau-\sigma)/l} \, \varepsilon_n^- \end{aligned}}$$

The same argument applies to the Fourier expansions of T_{++} and T_{--}:

$$\boxed{\begin{aligned} T_{++}(\sigma^+) &= \frac{2\pi}{l^2 T} \sum_m L_m^{(+)} \, e^{-2\pi i m(\tau+\sigma)/l} \\ T_{--}(\sigma^-) &= \frac{2\pi}{l^2 T} \sum_m L_m^{(-)} \, e^{-2\pi i m(\tau-\sigma)/l} \end{aligned}}$$

Using this in $Q_{(\pm)}$ one gets,

$$Q_{(+)} = T\int_0^l T_{++}\varepsilon^+ d\sigma = \frac{2\pi}{l^2}\sum_{m,n} L_m^{(+)}\varepsilon_n^{(+)} e^{-2\pi i(m+n)\tau/l}\underbrace{\int_0^l e^{-2\pi i(m+n)\sigma/l}d\sigma}_{l\,\delta_{m+n,0}}$$

$$\boxed{\begin{aligned} Q_{(+)} &= \frac{2\pi}{l}\sum_{m=-\infty}^{\infty} L_m^{(+)}\varepsilon_{-m}^{(+)} \\ Q_{(-)} &= \frac{2\pi}{l}\sum_{m=-\infty}^{\infty} L_m^{(-)}\varepsilon_{-m}^{(-)} \end{aligned}}$$

For each $\varepsilon_m^{(\pm)}$ we get the corresponding conserved charges $L_{-m}^{(\pm)}$ which are essentially the Fourier modes of T_{++} and T_{--} [Explicity, for example, for $\varepsilon^+ = \varepsilon_{-m}^{(+)} e^{-2\pi i m(\tau+\sigma)}$, $J_+ = TT_{++}\varepsilon^+ = T\varepsilon_{-m}^{(+)} e^{-2\pi i m(\tau+\sigma)}T_{++}$ and $\int J_+ d\sigma = L_m^{(+)}$]. One also has,

$$\boxed{\begin{aligned} L_m^{(+)} &= \frac{lT}{2\pi}\int_0^l T_{++}(\sigma^+)\, e^{+2\pi i m\sigma/l} d\sigma\Big|_{\tau=0} \\ L_m^{(-)} &= \frac{lT}{2\pi}\int_0^l T_{--}(\sigma^-)\, e^{-2\pi i m\sigma/l} d\sigma\Big|_{\tau=0} \end{aligned}}$$

$L_m^{(\pm)}$ are dimensionless.

In particular,

$$H = \frac{lT}{2\pi}\int_0^l (T_{++} + T_{--})\, d\sigma = \left(L_0^{(+)} + L_0^{(-)}\right)$$

$$P = \frac{lT}{2\pi}\int_0^l (T_{--} - T_{++})\, d\sigma = \left(L_0^{(-)} - L_0^{(+)}\right)$$

Note that $L_m^{(\pm)}$ are functionals of X^μ. To define Poisson brackets involving $L_m^{(\pm)}$ we express these in terms of X^μ and their canonically conjugate momenta Π_μ. For example,

$$L_m^{(+)} = \frac{1}{4}\left(\frac{lT}{2\pi}\right)\int_0^l d\sigma \left(\frac{\pi^2}{T^2} + \acute{X}^2 + \frac{2}{T}\,\Pi \cdot \acute{X}\right) e^{2\pi i m \sigma / l}$$

where $\Pi^2 = \Pi^\mu \Pi_\mu$, and $\Pi_\mu = \dfrac{\partial \mathcal{L}}{\partial \partial_0 X^\mu} = T\partial_0 X_\mu$.

Now we can easily compute the functional derivatives $\dfrac{\delta L_m^{(+)}}{\delta X^\nu}$ and $\dfrac{\delta L_m^{(+)}}{\delta \Pi_\nu}$ (For details see the next few pages)

using these, the Poisson bracket $\{L_m^{(+)}, L_n^{(+)}\}_{PB}$ can be easily computed as

$$\boxed{\{L_m^{(+)}, L_n^{(+)}\} = i\,(n-m)\,L_{m+n}^{(+)}}$$

with a similar expression for $L_m^{(-)}$. We also know that in general, when a theory is quantized, the Poisson brackets go over to commutators: $[\,,\,] = i\{\,,\,\}_{P.B}$. Then, naively, in the quantum theory we expect to have

$$[L_m, L_n] = (m-n)\,L_{m+n}$$

both for $L^{(+)}$ & $L^{(-)}$. This is the "Virasoro algebra with zero central charge". In reality, the quantum version of the algebra also develops a non-zero central extension as will be discussed later. The above infinite dimensional algebra is the symmetry algebra of classical string theory.

We will see later that in situations where boundary conditions (4) are not satisfied, one can still define conserved quantities as in (1) and satisfy boundary conditions (2) by choosing

$$\varepsilon_n^{(+)} = \varepsilon_n^{(-)} = \varepsilon_n$$

Then,

$$Q_\varepsilon = \frac{2\pi}{l} \sum_n \varepsilon_n \left(L_n^{(+)} + L_n^{(-)} \right)$$

and $\quad L_n^{(+)} = L_n^{(-)}$.

At classical level, the constraint $T_{\alpha\beta} = 0$ amounts to

$$L_m^{(+)} = 0, \qquad L_m^{(-)} = 0$$

On page 78 we discuss the L_n's in more detail.

Conserved Charges as Generators of Transformations:

Noether's theorem shows that the invariance of the action under a transformation leads to the existance of a conserved charge Q_ϵ. However, there is more to the relation between Q_ϵ and the corresponding transformation. Here, we show that Q_ϵ acts as the generator of the transformation in the sense that

$$\bar{\delta}_\epsilon \phi_a(x) = -\left\{ Q_\epsilon , \phi_a(x) \right\}_{P.B.}$$

where $\{ , \}_{P.B.}$ stands for the "Poisson bracket" and $\bar{\delta}_\epsilon \phi_a$ was defined in our discussion of Noether's theorem. But first, let us make a digression and discuss some concepts that will be needed for the proof:

Digression

(A) Functional Derivatives: Let $f_a(y)$ be functions in an n-dimensional space-time with coordinates y^α, $(\alpha = 0, 1, \ldots n-1)$. Consider a function $\mathcal{F}$ of the f_a and their first derivatives, $\mathcal{F} = \mathcal{F}(f_a(y), \partial_\alpha f_a(y))$

Then

$$F[f_a] = \int_\Omega d^n y \; \mathcal{F}(f_a(y), \partial_\alpha f_a(y))$$

is said to be a "functional" of f_a (it depends on the functional form of $f_a(y)$). Consider a local variation of $f_a(y)$:

$$f_a(y) \longrightarrow f_a(y) + \delta f_a(y)$$