Final Examination Paper for Mechanics (Fy2010)

Allowed help material: *Physics and Mathematics handbooks*

Date: *Wednesday, May 04, 2005*, Time: *09:00 - 15:00* [**Solutions**]

Questions:	1a	1b	1c	2a	3a	3b	4a	4b	5a	5b	6a	6b	Total
Marks:	5	2	3	10	7	3	6	4	5	5	6	4	60

1. (a) On a sheet of slippery ice, an eskimo of mass M_e pulls with constant force F on a rope of negligible mass the other end of which is attached to a sled of mass M_s. Initially, the sled is a distance D away from the eskimo and both start moving from rest. Ignoring friction, find the distance that the sled travels to the point where it meets the eskimo.

   (b) As soon as they meet, the eskimo holds tightly onto the sled. What is the total momentum of the combined system at this moment?

   (c) Now consider the above situation when ice has a friction coefficient μ. Find the total momentum of the system at the moment that the sled and the eskimo meet.

Solution:

a) As the eskimo pulls the sled with force F, the sled exerts a reaction force $-F$ on the eskimo. The magnitudes of their accelerations are $a_e = F/M_e$ and $a_s = F/M_s$ and in time t they travel distances $d_e = \frac{1}{2}a_e t^2$ and $d_s = \frac{1}{2}a_s t^2$. The time t_m at which the sled and the eskimo meet is found from $D = d_s + d_e$. Substituting back in d_s give the distance travelled by the sled when it meets the eskimo:

$$t_m = \sqrt{\frac{2D}{F}\frac{M_e M_s}{M_e + M_s}}, \qquad d_s = \frac{D M_e}{M_e + M_s}$$

b) At time t_m the magnitudes of the momenta are $p_e = M_e a_e t_m = F t_m$ and $p_s = M_s a_s t_m = F t_m$. Hence they are equal in magnitude but oppositely directed and therefore add up to zero total momentum. This also follows from $\Delta\vec{p}_{total} = \int \vec{F}_{total} dt$ where in our case $F_{total} = F - F = 0$.

c) In the presence of friction, the forces acting on the sled and the eskimo are $f_s = F - \mu g M_s$ and $f_e = F - \mu g M_e$ and are oppositely directed. The total force is then $F_{total} = f_e - f_s$ (pointing from the eskimo to the sled). The total initial momentum is zero so the total final momentum at time t'_m when they meet is given by

$$p_{final} = \int_0^{t'_m} F_{total} dt = \mu g (M_s - M_e) t'_m$$

The new meeting time t'_m can be calculated as in part a), now with the accelerations given by $a_e = f_e/M_e$ and $a_s = f_s/M_s$ so that $t'_m = \sqrt{\frac{2DM_eM_s}{F(M_e+M_s) - 2\mu g M_s M_e}}$.

2. (a) A rocket of mass M is fired upwards with initial velocity v_0. Find the distance R from the centre of the earth at which the rocket comes to rest before falling back. How will the hight R get affected if the direction of the initial velocity is changed by some small angle? Justify your answer.

Solution:
We use the conservation theorem for mechanical energy in the absence of dissipative forces, $K_f + U_f = K_i + U_i$. The initial and final kinetic energies are $K_i = \frac{1}{2}Mv_0^2$, $K_f = 0$ and for the potential energies, $U_i = -GM_eM/R_e$ and $U_f = -GM_eM/R$, where R_e and M_e denote the radius and mass of the earth, respectively, and $R = R_e + h$ is the final distance of the rocket from the centre of the earth (note the negative sign in U so that as you go away from the earth, U incearses by becoming less negative). Substituting the values we can solve for R or $h = R - R_e$ as

$$R = \frac{R_e}{1 - (v_0^2 R_e/2GM_e)}$$

Now consider the effect of a small change in the direction of $\vec{v}_0$. First, such a change will not affect K_i which depends only on v_0^2 and not on the direction of $\vec{v}_0$. Also, $U_f - U_i = -\int \vec{F} \cdot \vec{dl}$ and since the gravitational force is central i.e., $\vec{F} = F\hat{r}$, only the radial component of $\vec{dl} = \vec{v}dt$ will contribute to the expression. Changing the direction of $\vec{v}_0$ changes the angular component of $\vec{dl} = dr\hat{r} + rd\theta\hat{\theta}$ and hence does not affect the expression for change in potential energy ($\hat{r} \cdot \hat{\theta} = 0$). However, if $\vec{v}_0$ is not entirely along $\vec{r}$ then at the topmost point, the rocket will still have a horizontal velocity and $K_f = \frac{1}{2}Mv_\theta^2 \neq 0$, so the height R to which it can rise decreases.

3. (a) Consider a uniform cylinder of mass m, radius b and moment of inertia $I_0 = mb^2/2$, that starts from rest and rolls down an inclined plane without slipping. The plane makes an angle θ with the horizontal. Find the velocity v of the cylinder's centre of mass after it has moved by a distance L along the inclined plane. Ignore friction so that mechanical energy is conserved.

(b) Recalculate v if the surface of the inclined plane has rolling friction coefficient μ.

Solution:
a) The cylinder rolls down the inclined plane purely under the influence of gravity. As it moves a distance L along the incline, its vertical position changes by $h = L\sin\theta$ and its graviational potential energy reduces by $mgL\sin\theta$. This is converted into the kinetic energies of its centre of mass motion (moving with velocity v) and its rotation (with angular velocity $\omega = v/b$). In the absence of slipping, no energy is spent on sliding friction, and if we ignore rolling friction, then energy conservation gives,

$$mgL\sin\theta = \frac{1}{2}mv^2 + \frac{1}{2}I_0\omega^2$$

From this we obtain $v = 2\sqrt{(gL\sin\theta)/3}$.
b) In this case the cylinder also has to do work against rolling friction as it rolls down

which dissipates a part of its energy. The friction force $\mu N = \mu mg\cos\theta$ *acts parallel to the inclined plane and is directed upwards resulting in the work* $\mu mgL\cos\theta$*. The energy conservation equation then becomes,*

$$mgL\sin\theta = \frac{1}{2}mv^2 + \frac{1}{2}I_0\omega^2 + \mu mgL\cos\theta$$

which gives the final velocity as $v = 2\sqrt{gL(\sin\theta - \mu\cos\theta)/3}$.

4. (a) A rigid body rotates about a fixed axis along the z-direction. Consider this body to be made up of a large number of particles of masses m_i at positions $\vec{r}_i$ with respect to some origin. Starting from the expression for the angular momentum of a constituent particle, $\vec{L}(i) = \vec{r}_i \times m_i\vec{v}_i$, derive an expression for the angular momentum of the rigid body along the z-axis L_z in terms of its moment of inertia I and the angular velocity of the rotation, ω.

   (b) Also in the same set up, show that $d\vec{L}/dt = \tau$, where τ is the total torque.

   Solution:
   a) Choosing an arbitrary point on the axis of rotation as the origin, the position $\vec{r}_i$ *of a particle can be written as* $\vec{r}_i = \vec{r}_{||i} + \vec{\rho}_i$ *where* ρ_i *is the perpendicular distance of the particle from the axis of rotation and* $\vec{r}_{||i}$ *is the component of* $\vec{r}_i$ *parallel to the z axis. Then,* $\vec{L}(i) = (\vec{r}_{||i} + \vec{\rho}_i) \times m_i\vec{v}_i$*. Clearly,* $\vec{r}_{||i}$ *does not contribute to* L_z *while the rest of the expression contributes only to* L_z*. Since* $\vec{v}$ *is perpendicular to both* $\vec{\rho}_i$ *and the z axis, we have* $\vec{\rho}_i \times \vec{v}_i = \rho_i^2\omega\hat{z}$*. Therefore,*

$$L_z = \left(\sum_i \vec{L}(i)\right)_z = \sum_i m_i\rho_i^2\omega = I\omega$$

   where we have used the definition of moment of intertia, $I = \sum_i m_i\rho_i^2$.
   b) Using $\vec{L} = \sum_i \vec{L}(i)$ *we get,*

$$\frac{d\vec{L}}{dt} = \sum_i\left(\frac{d\vec{r}_i}{dt} \times m_i\vec{v}_i + \vec{r}_i \times m_i\frac{d\vec{v}_i}{dt}\right) = \sum_i\left(\vec{r}_i \times \vec{F}_i\right) = \sum_i \vec{\tau}_i = \vec{\tau}$$

   since $d\vec{r}_i/dt \times \vec{v}_i = \vec{v}_i \times \vec{v}_i = 0$

5. (a) Consider an undamped forced harmonic oscillator consisting of a mass m attached to a spring of spring constant k and subjected to a periodic deriving force $F_0\cos(\omega t)$. For oscillations of the type $x = A\cos(\omega t)$, determine the amplitude and discuss the phenomenon of "resonance".

   (b) In the above problem, assume that the deriving force is suddenly set to zero precisely at a moment that the oscillator is passing through its equilibrium point, for example, at time $t = \pi/2\omega$. Determine the subsequent motion of the system.

Solution:

a) Substituting the solution $x = A\cos\omega t$ *in the equation* $\ddot{x} + \omega_0^2\, x = \frac{F_0}{m}\cos\omega t$*, where* $\omega_0^2 = k/m$*, we obtain the expression for the oscillation amplitude,*

$$A = \frac{F_0}{m}\,\frac{1}{\omega_0^2 - \omega^2}$$

As the driving frequency ω *approaches the natural frequency of the oscillator* ω_0*, the oscillation amplitude grows very large, finally blowing up for* $\omega = \omega_0$*. This phenomenon is called resonance.*

b) For $t \geq \pi/2\omega$*, we have a simple harmonic oscillator* $\ddot{x} + \omega_0^2\, x = 0$*, with* $\omega_0^2 = k/m$*. It has a solution* $x_{SHO} = A_0\cos(\omega_0 t + \phi_0)$*. The constants* A_0 *and* ϕ_0 *are determined by the initial conditions on* x_{SHO} *and* $\dot{x}_{SHO}$ *at* $t = \pi/2\omega$*:*

$$x_{SHO}(t = \frac{\pi}{2\omega}) = x(t = \frac{\pi}{2\omega})\,, \qquad \dot{x}_{SHO}(t = \frac{\pi}{2\omega}) = \dot{x}(t = \frac{\pi}{2\omega})$$

This gives, $\phi_0 = \frac{\pi}{2}(1 - \frac{\omega_0}{\omega})$ *and* $A_0 = A\omega/\omega_0$*.*

6. (a) A rod of lenght L lies at rest in the xy plane, making an angle θ with the x-axis. Find the length $\widetilde{L}$ and orientation $\widetilde{\theta}$ of the rod as measured by an observer who is moving with constant velocity $-v$ along the x-axis.

   (b) How much energy is needed to boost the velocity of an electron of mass m_e from zero to half the velocity of light $c/2$?

Solution:

a) In the rest frame, the length L *has components in the* x *and* y *directions,* $L_x = L\cos\theta$*,* $L_y = L\sin\theta$*. In a frame moving along the* x *axis, the length* L_x *undergoes Lorentz contraction to* $\widetilde{L}_x = L_x\sqrt{1 - v^2/c^2}$ *while* $\widetilde{L}_y = L_y$ *is unchanged. Hence in the moving frame the length of the rod is given by* $\widetilde{L}^2 = \widetilde{L}_x^2 + \widetilde{L}_y^2$ *and the new angle is* $\tan\widetilde{\theta} = \widetilde{L}_y/\widetilde{L}_x$ *or,*

$$\widetilde{L} = L\sqrt{1 - \frac{v^2}{c^2}\cos\theta}\,, \qquad \tan\widetilde{\theta} = \gamma\tan\theta$$

b) The relativistic energy of an electron moving with velocity v *is given by* $E_v = m_e c^2/\sqrt{1 - v^2/c^2}$*. Thus the energy needed to boost its velocity from* $v = 0$ *to* $v = c/2$ *is*

$$\Delta E = E_{v=c/2} - E_{v=0} = (\frac{2}{\sqrt{3}} - 1)\, m_e c^2$$