



*Exploring Exotic Quantum Hall States Using
Exact Diagonalization*

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Abstract

We explore the quantum Hall system on a torus as a function of its circumference and the electron-electron interaction using exact diagonalization. The interaction is varied by the use of Haldane pseudopotentials to simulate the physics of different Landau levels and to investigate the stability of the various phases. Recent results on the thin torus show that it provides a simple understanding of established phenomena and also new results. Motivated by this we focus on the connection between the very thin torus and the bulk system. In particular, we investigate half-filled Landau levels, where the physics of the two lowest Landau levels are known to be quite different. In the lowest Landau level ($\nu = 1/2$), the system is well described by free composite fermions and is as such gapless. In the second Landau level ($\nu = 5/2$), the system is gapped and is believed to be described by the Pfaffian Moore-Read state. This state has quasiparticles/holes of charge $\pm e/4$ that obey non-Abelian statistics. The connection between this kind of mysterious statistics and the proposed possibility of using the system for decoherence free qubits in quantum computers makes this a very hot subject.

In this thesis we identify the two phases described above and study their properties. We find that the known sixfold degeneracy of the Pfaffian state on the torus is realized by six inequivalent crystalline states on the very thin torus. Furthermore, the $\pm e/4$ quasihole and quasiparticle excitations emerge as domain walls between these vacua. Our exact diagonalization studies show that these properties are robust as the circumference of the torus grows, indicating that the description on the thin torus is also relevant for the bulk system. Moreover, we verify that earlier results on the gapless $\nu = 1/2$ system are stable against perturbations of the interaction.

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Chapter 1

Introduction and outline

1.1 Introduction

The integer quantum Hall effect (IQHE) was first discovered in 1980 by von Klitzing¹, Dorda and Pepper [1]. Two years later, the even more interesting fractional quantum Hall effect (FQHE) was discovered by Tsui, Störmer and Gossard [2]. Since then it has been the object of interest for many research projects, but still the FQHE offers challenging mysteries not yet resolved.

The quantum Hall effect (QHE) arises in two-dimensional electron gases exposed to strong magnetic fields perpendicular to the surface. Classically, there is a linear relation between the Hall resistance in the sample and the strength of the magnetic field, B . However, as the temperature gets low, the magnetic strength grows large and the samples get cleaner, the resistance gets quantized. When the resistance is plotted as a function of B , plateaus emerge at the values $R_{xy} = \frac{R_K}{\nu}$. ν here is a fractional number and is called the filling fraction. The emergence of the plateaus is a nontrivial consequence of the interaction between the electrons.

In spite of the relatively simple experimental setup, the theory behind the FQHE contains several peculiar phenomena like e.g. fractional charge and statistics [3]. Behind some of the greatest theoretical progresses stand Laughlin [4] and Jain [5], who have e.g. constructed microscopical wave functions describing several quantum Hall systems². These commonly accepted theories describing the quantum Hall system are very successful, but not everything is based on a complete microscopical understanding of the problem.

Earlier work by E.J. Bergholtz and A. Karlhede [6], [7] has lead to inter-

¹von Klitzing was awarded the Nobel Prize five years later for his achievement.

²Laughlin shared a Nobel Prize with Tsui and Störmer.

esting results on generic filling fractions in the lowest Landau level when the system is put on a thin torus. In particular a microscopic derivation of composite fermions has been obtained in the $\nu = 1/2$ system on the thin torus. The conclusion is that much of the two-dimensional bulk physics remains in this limit. Encouraged by this, we study another half-filled system, $\nu = 5/2$, on the thin torus and how it is connected to the bulk system [8]. We find the ground states of this phase by varying the electron-electron interaction in the Hamiltonian of the $\nu = 1/2$ system. This is in turn realized by varying the expansion parameters of the so-called Haldane pseudopotentials.

The 'Pfaffian' state, which is believed to describe the $\nu = 5/2$ system, exposes several interesting and unusual features that motivate the title of this thesis. For example, fractional charges $\pm e/4$ appear in excitations of the degenerate ground states. The fractional charges (which we may call quasiparticles) obey non-Abelian statistics, that qualitatively differs from the statistics of fermions, bosons and anyons. This property has lead to an interesting discussion of whether the $\nu = 5/2$ system can be used in future quantum computing, and as such this is a subject within quantum physics that receives great attention. Previous successes of studying half-filled quantum Hall systems on the thin torus certainly encourages the investigation of the exotic Pfaffian system on this geometry.

1.2 Outline

In chapter 2 we give a short introduction to the classical as well as the quantum Hall effects. We also present a brief review of some of the basic theories behind the QHE and introduce the concept of fractional charges and fractional statistics.

In chapter 3 we consider electrons in a magnetic field on a cylinder. First we treat the non-interacting case and derive the single-particle wave functions and introduce the concept of Landau levels. This is then used to construct a one-dimensional lattice model, where each single-particle state is either occupied or empty, of the interacting two-dimensional many-particle system in a single Landau level. This mapping of the 2D quantum Hall system to a 1D lattice model is exact in the strong magnetic field limit. We generalize this to the geometry of a torus and construct a field operator Hamiltonian. We also give a rather comprehensive discussion of the various symmetries of the model. At the end of the chapter we discuss the thin limit of the model, i.e. when the circumference of the torus goes to zero.

The focus of this thesis is on the half-filled quantum Hall system, and in chapter 4 we present more of the theory and phenomena connected to

half-filling in the lowest ($\nu = 1/2$) and the second ($\nu = 5/2$) Landau levels. We argue that the $\nu = 5/2$ system can be treated as being in the lowest Landau level but with a different kind of interaction than Coulomb. We present the method of varying the interaction in the quantum Hall system using Haldane pseudopotentials. We write down formulas for changing the form of the interaction in Fourier space on the torus and discuss how this can be translated to a real space interaction.

In chapter 5 we briefly describe a computer program for exact diagonalization that has been used during this diploma work. We present a phase diagram for the half-filled quantum Hall system as a function of the pseudopotential parameter and the circumference of the torus. We identify the $\nu = 5/2$, or Pfaffian, phase and investigate its properties especially on the thin torus. This exposes a simple picture of the emergence of fractional charges. Results related to ours were obtained by Haldane [9] and, for a bosonic case, by Seidel and Lee [10].

The last chapter summarizes the results presented in the thesis and presents an outlook towards further research possibilities.

The main results of this diploma work are presented in an article [8] that was written together with E.J. Bergholtz, J. Kailasvuori, T.H. Hansson and A. Karlhede. This paper can be found in the appendix.

Chapter 2

The Hall experiments

2.1 The classical Hall effect

The Hall effect arises when a conducting plate is placed in a magnetic field \mathbf{B} perpendicular to the surface, and a current \mathbf{I} is driven through the plate (see Fig. 2.1). With the coordinate axes chosen as in the figure, $\mathbf{B} = B\hat{z}$ and $\mathbf{I} = I\hat{x}$.

If one considers the classical picture of this system, each electron is affected by the Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where the charge $q = -e = -|e|$. Since the velocity \mathbf{v} at equilibrium is antiparallel to the current and perpendicular to the magnetic field, the electrons will experience a force pointing in the negative y -direction. As the charged particles accumulate at one side of the metal plate, a potential difference builds up. In equilibrium the magnetic force will balance the electric repulsion between the electrons, and an easy calculation shows that the potential difference V_y becomes proportional to B . In 1879, the American physicist Edwin Hall [11] discovered this linear behavior when he plotted the so-called Hall resistance $R_H = V_y/I$ against the magnitude of the magnetic field—the result was, as we now expect, a straight line. This simple phenomenon is called the (classical) Hall effect.

2.2 The quantum Hall effect

About a century after Hall's achievement, in 1980, the quantum version of the Hall effect was discovered by Klaus von Klitzing and co-workers [1]¹. At very low temperature and in strong magnetic fields, electrons in thin interfaces between semi-conductors form a two-dimensional electron gas (i.e.,

¹An introduction to the physics of the quantum Hall effect is found in [12].

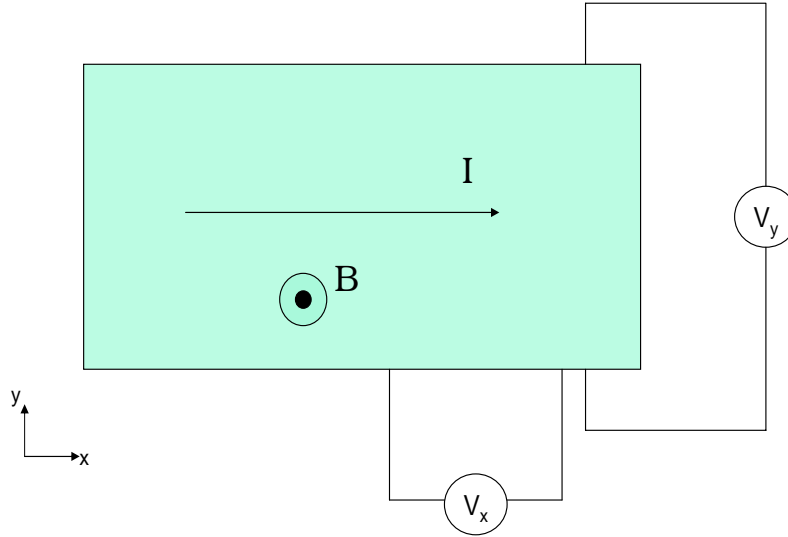


Figure 2.1: *The setup for the Hall experiment.*

they are unable to move in the z -direction). Under these conditions, and if the sample has a small (but crucial) amount of disorder, the previously linear dependance of B is destroyed. Instead, a quantization of the Hall resistance appears. von Klitzing observed that when the magnetic field strength varies, the Hall resistance $R_{xy} = V_y/I$ jumps between plateaus of specific values (see Fig. 2.2), namely

$$R_{xy} = \frac{R_K}{n}, \quad n = 1, 2, 3, \dots, \quad (2.1)$$

where $R_K = h/e^2$ is the so called von Klitzing constant. Note also from the figure that, as R_{xy} forms plateaus, the longitudinal resistance $R_{xx} = V_x/I$ drops to zero. Because the denominator n is an integer, this behavior of the system is now called the integer quantum Hall effect. It can be explained by first solving the quantum mechanical problem of an electron in a magnetic field (see section 3.1).

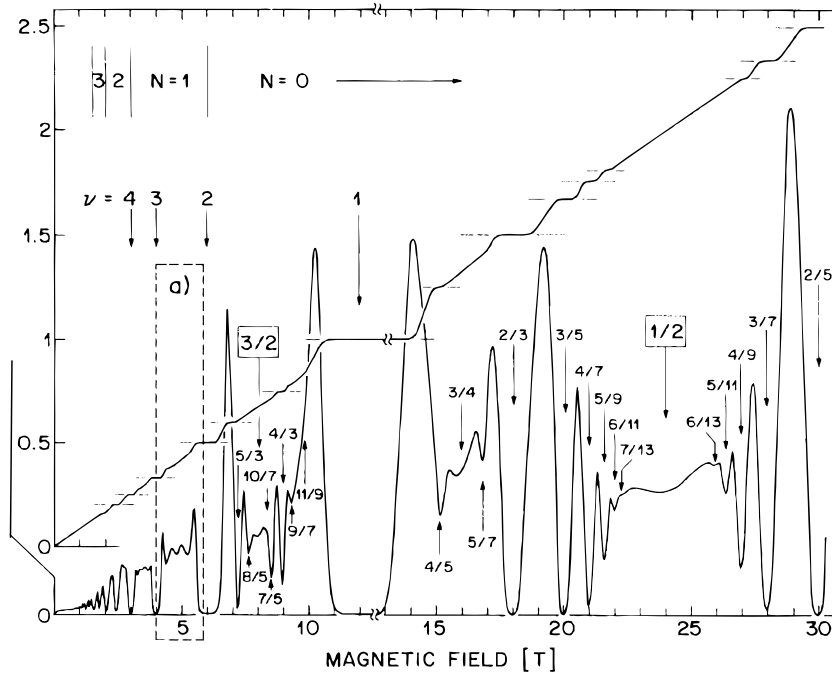


Figure 2.2: The resistances R_{xy} (the nearly straight line with plateaus) and R_{xx} (roller coaster curve) as a function of the magnetic field. The figure is borrowed from Willett et al. [13].

However, there is also a similar, but much harder-to-explain, quantum effect which arises at even more extreme experimental conditions. For cleaner samples and as the temperature is decreased further, a new set of plateaus can be found at some 'magical' values

$$R_{xy} = \frac{R_K}{\nu}, \quad (2.2)$$

where ν is a fractional number; $\nu = \frac{p}{q}$, where q typically is an odd integer (again, see Fig. 2.2)—this is the fractional quantum Hall effect². It was first observed by Tsui, Störmer and Gossard in 1982 [2] at $\nu = 1/3$. Many of these plateaus can be seen in Fig. 2.2. What also can be seen in the figure, is that at some fractions, e.g. $\nu = 1/2$, R_{xy} is not quantized. At these points, R_{xx} has small dips but does not go to zero. These different behaviors at different ν -values point at very different and interesting physics (see the next section).

²For reasons that will become clear in the next section, ν is called the filling fraction of the system.

2.3 Basic theory

As mentioned earlier, the IQHE can be understood by investigating the quantum mechanical properties of a single electron in a magnetic field. It turns out that the degenerate energy levels of such an electron are those of a harmonic oscillator, i.e.

$$E_n = (n + 1/2)\hbar\omega_c, \quad n = 0, 1, \dots, \quad (2.3)$$

where $\omega_c \equiv \frac{eB}{mc}$ is the cyclotron frequency. These energy levels are called Landau levels, after the Russian physicist L.D. Landau (1908-1968) [14] who was the first to solve this problem. In the low temperature limit (i.e. the quantum Hall regime), the electrons in the two-dimensional electron gas will simply occupy the states with the lowest energies, if electron-electron interaction is ignored. Each Landau level has a degeneracy that depends on the strength of the magnetic field. The number of states N_S within each energy level is

$$N_S = \frac{BA}{\Phi_0}, \quad (2.4)$$

where A is the area of the Hall plate and Φ_0 is the magnetic flux quantum, $\Phi_0 = \frac{hc}{e}$. In other words, N_S is given by the number of flux quanta penetrating the system. We may now define a very important quantity, namely the filling fraction ν ;

$$\nu = \frac{N_e}{N_S} = \frac{N_e \Phi_0}{BA}, \quad (2.5)$$

where N_e is the number of electrons in the sample. If we let the electrons fill the lowest available single-particle states, ν will be the number of filled Landau levels. The ν that appears in equation (2.2) is closely related to the filling fraction. The plateau with $R_{xy} = \frac{R_K}{\nu}$ is namely centered around the value $B = \frac{N_e \Phi_0}{\nu A}$, and we can now comment on the emergence of the integer effect. When ν is an integer, the ν lowest lying Landau levels will be completely filled and there is an energy gap to the next level. In presence of disorder in the system, this energy gap will lead to a plateau in the resistance, i.e. changing the magnetic field around these points will keep R_{xy} constant. The detailed theory explaining this is a bit complicated, and will not be a subject of this thesis. The interested reader can read about Laughlin's so-called gauge argument in [15].

In general, the requirements for the quantum Hall effect to appear, besides low temperature and high magnetic fields, are some finite amount of disorder in the system, and a finite energy gap. Disorder is of course always present in the experimental situation.

The difficulty in understanding the fractional effect lies in the fact that the filling fraction in this case does not cause an energy gap by itself. Since the highest occupied Landau level is not completely filled for fractional ν 's, there are many ways of arranging the electrons within that energy level. If one considers interaction between the electrons, excitations within the highest occupied Landau level are possible. For some filling fractions, the interaction is of the kind that it leads to a gap, whereas some interactions lead to a continuous energy spectrum within the highest occupied Landau level. Indeed, for $\nu = 1/2$ and some other fractions, the resistance is not quantized (i.e. no gap!), but there are many examples of fractional fillings which display plateaus in the resistance curve, as seen in Fig. 2.2. Clearly the interaction between the electrons, which could be neglected when considering the IQHE, plays a crucial role in the FQHE and render a gap that will give rise to a quantization of R_{xy} . At these fillings the many-body ground state must have a gap in energy to the excited states.

The questions to be answered are: In what way do the electrons interact at various fractional fillings, and why do their wave functions, for some fractions but not for others, give rise to an energy gap? Several attempts have been made to solve these mysteries. Most important are Laughlin's wavefunction for $\nu = \frac{1}{2m+1}$ [4] and the wave function for $\nu = \frac{p}{2pm+1}$, given by Jain [5]. Here, both m and p are integers, which gives odd denominator fillings.

In the plane, the unnormalized Laughlin wave function reads

$$\Psi_{\frac{1}{2m+1}}(z_1, z_2, \dots, z_{N_e}) = \prod_{i < j} (z_i - z_j)^{2m+1} e^{-\frac{1}{4} \sum_i |z_i|^2}, \quad (2.6)$$

where $z = x + iy$ and (x, y) are the particle coordinates in the plane. The numbers 1 to N_e label the electrons. Note that the polynomial factor leads to an antisymmetric wavefunction (fermions!) and that it makes the total function rapidly shrink to zero for electrons close to each-other. This trial wavefunction is an extremely good approximation to the ground state at $\nu = \frac{1}{2m+1}$, and it predicts both the expected gap and excitations with fractional charge $\pm \frac{e}{2m+1}$.

Jain's wavefunction is a good approximation for $\nu = \frac{p}{2mp+1}$ and reads

$$\Psi_{\frac{p}{2mp+1}}(z_1, z_2, \dots, z_{N_e}) = \mathcal{P}_{LLL} \prod_{i < j} (z_i - z_j)^{2m} \Psi_p, \quad (2.7)$$

where Ψ_p is the state with the p lowest Landau levels completely filled and \mathcal{P}_{LLL} projects the entire state on the lowest Landau level. Jain also introduced so-called composite fermions to explain the FQHE. Composite fermions are electrons 'bound to' an even number of flux quanta, which so-to-say reduces the magnetic field and thus increases the filling fraction (see

Fig. 2.3). More specifically, the 'new' magnetic field becomes

$$B^* = B - 2m\nu B. \quad (2.8)$$

In that way, Jain proposes, one can interpret the FQHE at e.g. $\nu = 1/3$ as a system of composite fermions at $\nu = 1$. In other words, we may see it as the IQHE but with a new kind of particles. Another element of this construction is that the composite fermions are non-interacting.

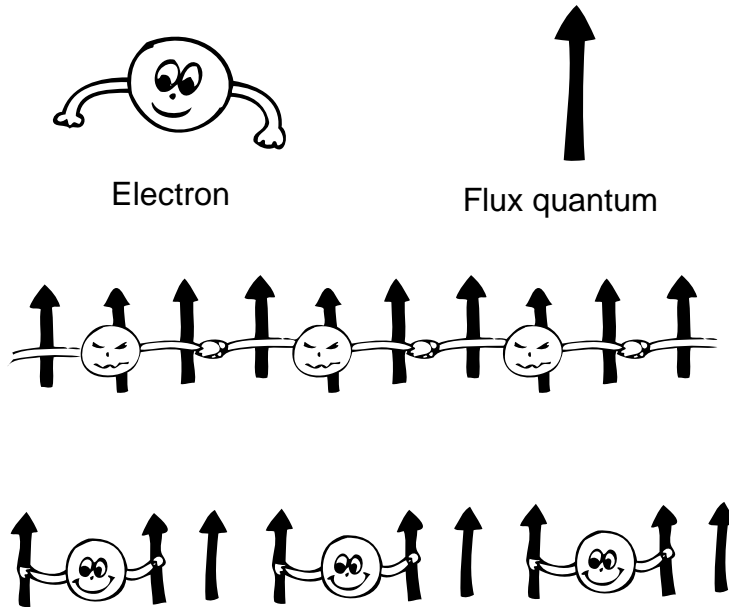


Figure 2.3: *An illustration of the composite fermion construction at $\nu = 1/3$. The strongly interacting electrons are seen as free composite fermions in a reduced magnetic field. The idea of the caricature is taken from Kwon Park.*

The strength of the composite fermions is that they describe the system at different filling fractions to a very high accuracy, but they have also been criticized for not providing a good microscopical understanding as a basis (see e.g. [16]) and basic questions remain to be answered. It is therefore an interesting challenge to explore the fractional quantum Hall effect in more detail. In chapter 4 there is a longer discussion concerning $\nu = 1/2$ and $\nu = 5/2$.

2.3.1 Fractional charges and statistics

The FQHE is a topic that contains many unusual phenomena. In this thesis we will among other things treat the appearance of fractional charges. Just

as it sounds, it means that there are excitations that behave like particles with fractions of an electron charge. In general, the gapped Jain states at $\nu = \frac{p}{2mp+1}$ have excitations with fractional charges $\pm \frac{e}{2mp+1}$. This will be further discussed in section 3.3.2.

Even more exotic phenomena are fractional and non-Abelian statistics, which differ from the statistics of bosons and fermions. Let us consider two indistinguishable particles, and think about what must happen if we let one of the particles go around the other particle and then return to its initial position. We require the operation to be adiabatic, i.e. we require finite gaps between the energy levels and move the particle so slowly that there is no energy transferred to the system. Suppose that the system initially is in the ground state. The adiabatic process then insures that the system is not excited to some higher energy level during the operation.

In three dimensions, there is no unambiguous definition of encircling a specific point in space, and the path can be contracted into a point. But this is the same thing as if nothing has happened and thus this operation must return the same state. Furthermore, it yields the same result as letting the two particles switch positions twice. Just interchanging the two particles clearly corresponds to half of the operation above, and should thus return the initial state with a factor of either plus or minus one in front. Fermionic (bosonic) statistics means that, because we are dealing with indistinguishable particles which can not (can) be in the same state, we must get a minus (plus) sign in front of the wave function when we let the two particles interchange. In other words, the wave function is antisymmetric (symmetric).

More possibilities appear when we consider the same operation in two dimensions. In that case there is really a way of defining what it means to go around a specific point. We may not shrink the path to a point, and can thus not say that nothing has happened. Clearly, if there is no degeneracy of the ground state, we must return to the same physical state after the operation. There is however a possibility of the original state being multiplied by a phase factor. Half of the operation, i.e. interchanging the two particles then also yields a phase factor $e^{i\phi}$. This kind of statistics is called fractional statistics [3]. The particles obeying this are called anyons and appear as quasiparticles in the Laughlin and Jain states. It is possible, but not yet established, that this behavior was observed experimentally in 2005 [17].

A generalization of fractional statistics is non-Abelian statistics, which means that interchanging two particles (so-called braiding) yields an entirely new state. This phenomenon appears when there is a degeneracy of the states. Since there is no energy transfer needed to put the system in the degenerate state, the adiabatic process does not prevent the particles from switching to the other states. Non-Abelian and fractional statistics are very

hot research subjects, both because of their exotic properties and, in the former case, since it could possibly play an important role in future quantum computing [18]. The presence of degeneracy is fulfilled for the quantum Hall system at e.g. $\nu = 5/2$, and a recent paper [19] has suggested a way to detect non-Abelian statistics experimentally for that specific case.

Chapter 3

The one-dimensional model

In this chapter we solve the quantum mechanical problem of a single electron on a cylinder with a magnetic field perpendicular to its surface. We use this to find an expression for the many-particle Hamiltonian on the cylinder and then on a torus. This leads to a mapping of the two-dimensional system onto a one-dimensional lattice. We also show how letting the circumference of the torus go to zero provides a nice way to study the quantum Hall system, and argue that many features of the large system remain in that thin limit.

3.1 Electron in a magnetic field

Since the quantum Hall effect takes place in a magnetic field perpendicular to a two-dimensional electron layer, a reasonable thing to do is to calculate the energy eigenstates for one of these electrons. We do this using the geometry of a cylinder and then generalize our results to the electron gas on a torus.

We want to study the situation in Fig. 3.1, where L_1 is the circumference of the cylinder. We are free to choose a gauge, and the so-called Landau gauge,

$$\mathbf{A} = By\hat{x}, \quad (3.1)$$

gives us a magnetic field pointing in the negative z -direction since $\mathbf{B} = \nabla \times \mathbf{A} = -B\hat{z}$.

As usual, we may start with the Hamiltonian of a free particle in two dimensions and modify this by minimal coupling:

$$\mathbf{p} \rightarrow \mathbf{p} - q\mathbf{A}, \quad (3.2)$$

which yields

$$H = \frac{1}{2m} \left(\left(p_x + \frac{eB}{c}y \right)^2 + p_y^2 \right). \quad (3.3)$$

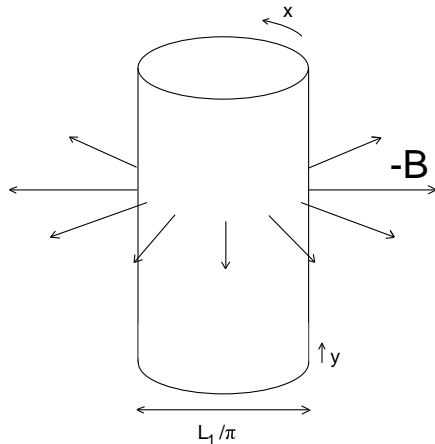


Figure 3.1: *An illustration of a cylinder with a magnetic field \mathbf{B} perpendicular to the surface.*

The question is: What are the energy eigenfunctions of this Hamiltonian?

A good choice of separating ansatz is

$$\psi_k(x, y) = e^{ikx} \phi(y). \quad (3.4)$$

Also, we should have periodic boundary conditions in the x -direction, i.e.

$$\psi_k(x, y) = \psi_k(x + L_1, y), \quad (3.5)$$

which implies

$$k = \frac{2\pi q}{L_1}, \quad q = 0, \pm 1, \dots \quad (3.6)$$

The wave function (3.4) is clearly an eigenstate of p_x with the eigenvalue $\hbar k$. This means that we are allowed to replace the p_x operator by $\hbar k$ in the Hamiltonian, and the x -dependence is eliminated. We can then divide both sides in the Schrödinger equation by e^{ikx} , and we are left with

$$H_k \phi(y) = E_k \phi(y), \quad (3.7)$$

with

$$H_k \equiv \frac{1}{2m} (p_y^2 + (\hbar k + \frac{eB}{c} y)^2). \quad (3.8)$$

We recognize this as the Hamiltonian of a one-dimensional harmonic oscillator, and can immediately write down the energy eigenvalues:

$$E_{nk} = (n + 1/2) \hbar \omega_c, \quad n = 0, 1, \dots, \quad (3.9)$$

where $\omega_c \equiv \frac{eB}{mc}$ is the cyclotron frequency. As mentioned in chapter 2.1, these energy levels are called Landau levels.

Note also that, according to eq. (3.8), the wave functions are centered around $y = -kl^2$, where $l \equiv \sqrt{\frac{\hbar c}{eB}}$ is the so-called magnetic length. With this definition, the energy eigenfunctions become

$$\psi_{nk}(x, y) = \frac{1}{\sqrt{\pi^{1/2} 2^n n! L_1}} e^{ikx} H_n(y + kl^2) e^{-\frac{1}{2l^2}(y+kl^2)^2}, \quad (3.10)$$

where H_n are the Hermite polynomials with $H_0 = 1$, $H_1(\xi) = 2\xi$, $H_2(\xi) = 4\xi^2 - 2$ etc..

Since the difference between two consecutive k -values is $\frac{2\pi}{L_1}$, the distance between the center of the associated wave functions is $\frac{2\pi l^2}{L_1}$, as illustrated by Fig. 3.2. In the following section this will be used to form a lattice model for the quantum Hall system, but first we would like to again identify the filling fraction ν . The area per state in a Landau level is $2\pi l^2$ (see Fig. 3.2), so the density of states is $n_s = 1/2\pi l^2$. If we let n_e be the electron density, the number of filled Landau levels is

$$\nu = \frac{n_e}{n_s} = 2\pi l^2 n_e = \frac{\hbar c n_e}{eB} = \frac{n_e \Phi_0}{B}, \quad (3.11)$$

which agrees with equation (2.5).

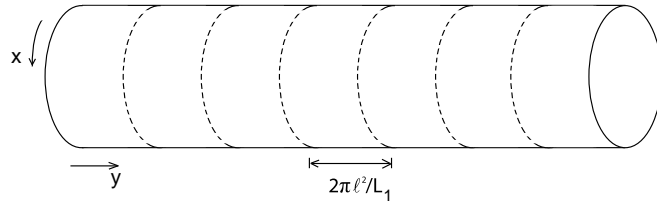


Figure 3.2: *The single particle states are centered along lines passing through the points $y = -kl^2$, a distance $2\pi l^2 / L_1$ apart.*

Through the rest of this work we let $l = 1$, i.e. the area per state is 2π . With L_2 being the length (in the y -direction) of the cylinder, this leads to the relation $L_1 L_2 = 2\pi N_s$.

3.2 A lattice model

We now turn to considering the full quantum Hall problem, i.e. the interacting many-particle system. We restrict our studies to the lowest Landau level ($n = 0$), assuming that the electrons will occupy only the states with the lowest energies¹. This presumption will be true in the limit $B \rightarrow \infty$, where the gap between the Landau levels grows large. That all electrons are in the lowest Landau level means that they have the same kinetic energy. The sum of these energies is a constant that can be subtracted from the Hamiltonian, which we may then regard as consisting of the potential of the electron-electron interaction only². Finally, we assume that we have complete spin-polarization—the large magnetic field will force all the electrons to have the same spin.

3.2.1 Fock space representation and symmetries

With these approximations we may consider many-particle states where each single-particle state $\psi_k(x, y)$ is either occupied by an electron or empty³. The single-particle states are centered along lines (see Fig. 3.2) that one can picture as sites in a one-dimensional lattice. Hence, we can perform a mapping onto a series of zeros and ones (Fock space representation), where a one at the k :th position denotes an electron in the state centered at $y = -k$ in units of the lattice spacing $2\pi/L_1$ (see Fig. 3.3)⁴. In other words, a one at this position corresponds to an electron with x -momentum $-\frac{2\pi}{L_1}k$ and k is here an integer. The correspondence between states in the lattice model and the single-particle states is, since we are dealing with fermions, a Slater determinant⁵:

$$|n_0 \ n_1 \dots \ n_{N_s-1}\rangle \doteq \frac{1}{\sqrt{N_e!}} \begin{vmatrix} \psi_{k_1}(\mathbf{r}_1) & \psi_{k_1}(\mathbf{r}_2) & \dots & \psi_{k_1}(\mathbf{r}_{N_e}) \\ \psi_{k_2}(\mathbf{r}_1) & \psi_{k_2}(\mathbf{r}_2) & \dots & \psi_{k_2}(\mathbf{r}_{N_e}) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{k_{N_e}}(\mathbf{r}_1) & \psi_{k_{N_e}}(\mathbf{r}_2) & \dots & \psi_{k_{N_e}}(\mathbf{r}_{N_e}) \end{vmatrix}, \quad (3.12)$$

¹Later we will also consider $\nu = 5/2$, which is in the second Landau level. However, we will treat even this case as belonging to the lowest level, but with modified electron-electron interaction.

²Interaction with the disorder in the system is ignored.

³Here, $\psi_k(x, y) = \psi_{0k}(x, y)$, i.e. the wave functions of the lowest Landau level. We could also have chosen to picture the system in e.g. the second Landau level, in which case we would have $\psi_k(x, y) = \psi_{1k}(x, y)$.

⁴The first lattice site to the left here corresponds to $k = 0$.

⁵Below, \doteq means that the state is *represented* by the wave function in position space of the particles, i.e. $|n\rangle \doteq \psi(\mathbf{r})$ means $\langle \mathbf{r}|n\rangle = \psi(\mathbf{r})$ etc.

where $n_i = 0, 1$ and $\sum_{i=0}^{N_s-1} n_i = N_e$. N_s is as before the number of states in the Landau level. A general many-particle state with N_e electrons is then constructed as a superposition of these states.

To make this clear, let us look at a system of just two electrons at filling $\nu = 1/2$. Suppose that one of them has zero momentum and one of them has a momentum corresponding to $k = 3$. The total wavefunction is then

$$\begin{aligned} |1001\rangle &\doteq \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_0(\mathbf{r}_1) & \psi_0(\mathbf{r}_2) \\ \psi_3(\mathbf{r}_1) & \psi_3(\mathbf{r}_2) \end{vmatrix} = \\ &= \frac{1}{\sqrt{2}} [\psi_0(\mathbf{r}_1)\psi_3(\mathbf{r}_2) - \psi_0(\mathbf{r}_2)\psi_3(\mathbf{r}_1)]. \end{aligned} \quad (3.13)$$

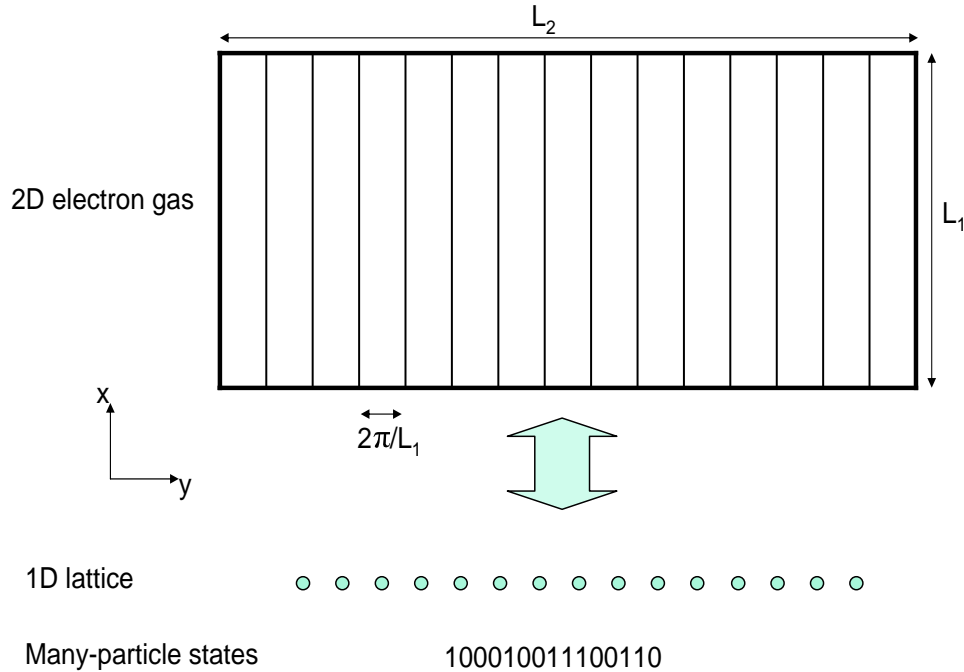


Figure 3.3: An illustration of the mapping between the single-particle states centered around different positions in a one-dimensional lattice. The many-particle basis states can be written as series of zeros and ones—the illustration shows one example of this.

Following Haldane [20], we will now introduce two translation operators, T_1 and T_2 , that act on the many-particle states described above. Consider a torus, i.e. our cylinder but with the ends connected so that the first and last sites are separated one lattice constant. This entails that we may require

periodic boundary conditions also in the y -direction⁶. We define T_2 to be an operator that translates the entire lattice configuration one step to the right (i.e. in the positive y -direction);

$$T_2|n_0 n_1 \dots n_{N_s-1}\rangle = |n_{N_s-1} n_0 \dots n_{N_s-2}\rangle. \quad (3.14)$$

T_1 is a translation operator acting in the x -direction, basically it picks out the x -momenta of the electrons, and the eigenvalues are $e^{\frac{i2\pi}{N_s} \sum_{k=0}^{N_s-1} kn_k} \equiv e^{i2\pi K/N_s}$. The lattice states of eq. (3.12) are eigenstates of T_1 ;

$$T_1|n_0 n_1 \dots n_{N_s-1}\rangle = e^{i2\pi K/N_s} |n_0 n_1 \dots n_{N_s-1}\rangle, \quad (3.15)$$

where

$$K = \sum_{k=0}^{N_s-1} kn_k \text{ mod } N_s. \quad (3.16)$$

K is thus the sum of the momenta (in units of $2\pi/L_1$) of the particles in the lattice, taken modulo N_s . This quantum number characterizes the eigenstates of T_1 . In the two-particle state described above, $K = 0 + 3 = 3$.

Because of the translation invariance on the torus, both T_1 and T_2 commute with the Hamiltonian in one Landau level, i.e. $[T_1, H] = [T_2, H] = 0$. However, $[T_1, T_2] \neq 0$. But let us now consider the electron gas at filling $\nu = p/q = N_e/N_s \Leftrightarrow pN_s = qN_e$. What happens if we let T_2^q act on one of the lattice states (which are eigenstates of T_1)? Each electron will move q steps to the right, increasing K with $N_e q$. Some of them, though, might at the same time 'fall over the edge' of the lattice, appearing at the left end again (remember the torus symmetry!). This will decrease K by mN_s , where m is the number of electrons falling over the edge. Summarizing, acting with T_2^q on an eigenstate of T_1 gives a change in K that is $\Delta K = N_e q - mN_s = (p - m)N_s$. Because p and m are integers, taking $\Delta K \text{ mod } N_s$ gives zero. This means that the operator T_2^q conserves the quantum number of T_1 and hence that $[T_2^q, T_1] = 0$. Since $[T_2, H] = 0$ it also follows that $[T_2^q, H] = 0$.

The operators T_1 and T_2^q form a maximal set of commuting operators together with the Hamiltonian, H . The three operators are simultaneously diagonalizable, and their common eigenstates constitute a complete basis set. This fact makes the procedure of diagonalizing the Hamiltonian easier and faster if we let the wanted energy eigenstates be eigenstates of T_1 and T_2^q as well. H will conserve the quantum numbers of T_1 and T_2^q , and so the

⁶One can of course also choose to have the wave function multiplied by an arbitrary phase factor without changing the physics.

matrix representation of H in this basis will only have non-zero elements for coupling between eigenstates with the same quantum numbers.

Since acting with T_2 on a state with N_s lattice sites N_s times must give back the original state, we have $T_2^{N_s} = 1$, i.e. the eigenvalue of $T_2^{N_s}$ is 1. We are interested in the eigenvalues of T_2^q . We rewrite $1 = T_2^{N_s} = (T_2^q)^{N_s/q}$, which implies that the eigenvalues of T_2^q are $a_N = e^{i2\pi Nq/N_s}$, where $N = 0, 1, \dots, \frac{N_s}{q} - 1$. N is thus the quantum number of T_2^q .

Let us now construct the eigenstates of T_2^q . Since we also search for eigenstates of T_1 , we require that each state in these combinations have the same K -value. This will however cause no trouble if we follow the following procedure. For each eigenvalue a_N , start with one of the states $|n_0 n_1 \dots n_{N_s-1}\rangle \equiv |\tilde{\Psi}\rangle$ and form the (unnormalized) state

$$|\Psi\rangle \equiv (1 + a_N^{-1}T_2^q + a_N^{-2}T_2^{2q} + \dots + a_N^{-(N_s/q-1)}T_2^{q(N_s/q-1)})|\tilde{\Psi}\rangle. \quad (3.17)$$

One can easily show that these states will be eigenstates of T_2^q , i.e. $T_2^q|\Psi\rangle = a_N|\Psi\rangle$. Some combinations will however give a trivial zero, in which case we have tried to match the eigenvalue with the wrong state. This method can clearly be used for any $\nu = p/q$ and N_s .

Here it is appropriate to comment on the degeneracy of the energy eigenstates. Since H and T_2 commute, it follows that translation between 1 and $q-1$ steps in the y -direction of a state yields new states with the same energy;

$$H|\Psi\rangle = E|\Psi\rangle \Rightarrow$$

$$HT_2|\Psi\rangle = ET_2|\Psi\rangle,$$

$$HT_2^2|\Psi\rangle = ET_2^2|\Psi\rangle,$$

\vdots

$$HT_2^{q-1}|\Psi\rangle = ET_2^{q-1}|\Psi\rangle. \quad (3.18)$$

Acting once more with T_2 on $|\Psi\rangle$ will return the first equation, since $|\Psi\rangle$ is an eigenstate of T_2^q as well. Note that all these translated states have different K -values, hence they are orthogonal. Thus we have shown that there is a q -fold degeneracy of the many-particle energies at filling $\nu = p/q$. This was first pointed out by Haldane [20].

Example

In this thesis, we study the case $\nu = 1/2$, in which each state is labeled by the quantum numbers K and N of T_1 and T_2^2 respectively. One task is thus to find the combinations that are eigenstates of T_1 and T_2^2 . Since we search for eigenstates of T_1 , we require that each state in these combinations have the same K -value; thus, for example, $|1100\rangle + |1010\rangle$ is not allowed since $|1100\rangle$ has $K = 1$ and $|1010\rangle$ has $K = 2$.

Again, let us look at the simple example of two electrons at four sites. First note that the number of ways to arrange two identical particles at four different sites is $\binom{4}{2} = 6$. We list the different possibilities and their K -values here:

$$\begin{aligned} &|1100\rangle; K = 1, \\ &|1010\rangle; K = 2, \\ &|1001\rangle; K = 3, \\ &|0110\rangle; K = 3, \\ &|0101\rangle; K = 4 \text{ mod } 4 = 0, \\ &|0011\rangle; K = 5 \text{ mod } 4 = 1. \end{aligned}$$

The second state above is the only one with $K = 2$, hence it can not be connected to any of the other states. We note that it is the eigenstate of T_2^2 with quantum number $N = 0$, since $T_2^2|1010\rangle = |1010\rangle = e^{i2\pi \cdot 0/2}|1010\rangle$. A similar thing holds for the fifth state, which has $K = 0$ and $N = 0$.

What about the four remaining states? Two by two they share the same K -value, but neither of them is alone an eigenstate of T_2^2 . If we do not immediately see the solution, we may use the general method described above. Let us for example try with $a_N = a_1 = -1$ and $|\tilde{\Psi}\rangle = |1001\rangle$:

$$|\Psi\rangle = \left(1 + \frac{1}{-1}T_2^2\right) |\tilde{\Psi}\rangle = |1001\rangle - |0110\rangle \quad (3.19)$$

so that

$$T_2^2|\Psi\rangle = |0110\rangle - |1001\rangle = -|\Psi\rangle = a_1|\Psi\rangle. \quad (3.20)$$

Thus, $|1001\rangle - |0110\rangle$ has $(K, N) = (3, 1)$, and so on. This and the rest of the results are summarized below:

$$\begin{aligned} &|1010\rangle; (K, N) = (2, 0), \\ &|0101\rangle; (K, N) = (0, 0), \\ &|1100\rangle + |0011\rangle; (K, N) = (1, 0), \\ &|1100\rangle - |0011\rangle; (K, N) = (1, 1), \end{aligned}$$

$$\begin{aligned} &|1001\rangle + |0110\rangle; (K, N) = (3, 0), \\ &|1001\rangle - |0110\rangle; (K, N) = (3, 1). \end{aligned}$$

For $\nu = 1/2$, the degeneracy of the energy eigenvalues is 2, and the degenerate states are translations one step in the y -direction of each-other. We check this by noting that e.g. $|1010\rangle$ and $|0101\rangle$ are related by this kind of translation. Also, it is evident from eq. (3.18) that they have the same energy.

3.2.2 Field operator Hamiltonian

The just described lattice model provides a simple way to construct the Hamiltonian for the electron-electron interactions, and (at least in the thin limit, as will be clear in the next section) explore the eigenstates of that operator.

Let us define a field operator $\hat{\Psi}^\dagger(\mathbf{r})$ that creates an electron at the position \mathbf{r} :

$$\hat{\Psi}^\dagger(\mathbf{r}) \equiv \sum_k \psi_k^*(\mathbf{r}) c_k^\dagger, \quad \{c_n^\dagger, c_m\} = \delta_{mn}, \quad (3.21)$$

where c_k^\dagger (c_k) is an operator that creates (destroys) an electron in the state ψ_k . One can then write the electron density as $\hat{\rho}(\mathbf{r}) = \hat{\Psi}^\dagger(\mathbf{r})\hat{\Psi}(\mathbf{r})$ and the Hamiltonian for the interaction energy between the electrons is⁷

$$\begin{aligned} H &= \frac{1}{2} \int \int : \hat{\rho}(\mathbf{r}_1) V(\mathbf{r}_1 - \mathbf{r}_2) \hat{\rho}(\mathbf{r}_2) : d^2 r_1 d^2 r_2 = \\ &= \frac{1}{2} \sum_{k_1 k_2 k_3 k_4} V_{k_1 k_2 k_3 k_4} c_{k_1}^\dagger c_{k_2}^\dagger c_{k_3} c_{k_4} \end{aligned} \quad (3.22)$$

where the matrix elements are

$$V_{k_1 k_2 k_3 k_4} = \int \int \psi_{k_1}^*(\mathbf{r}_1) \psi_{k_2}^*(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2) \psi_{k_3}(\mathbf{r}_2) \psi_{k_4}(\mathbf{r}_1) d^2 r_1 d^2 r_2. \quad (3.23)$$

The symbol $::$ means that the product of operators is normal ordered, i.e. all creation operators are put to the left of the annihilation operators (this is a way of preventing the empty lattice from yielding non-zero energy terms).

The sum in eq. (3.22) contains two kinds of electrostatic terms. For $k_1 = k_4$, $k_2 = k_3$ particle 1 and 2 keep their original k -values (compare with eq. (3.23))—this is called the direct interaction. There is also the possibility for $k_1 = k_3$, $k_2 = k_4$, which corresponds to particle 1 and 2 switching places

⁷Compare with the classical expression $H_{cl} = \frac{1}{2} \int \int \rho(\mathbf{r}_1) V(\mathbf{r}_1 - \mathbf{r}_2) \rho(\mathbf{r}_2) d^2 r_1 d^2 r_2$.

in the lattice—this is referred to as the exchange interaction. Remember that the electrons are indistinguishable, so that the hop does not change the state. We will therefore refer to these terms as electrostatic. The Hamiltonian also contains so-called hopping terms, where the above conditions for k_i are not fulfilled. For both cases however, the total momentum of the particles must be conserved—or, the position of the center of mass must be preserved. Since the different sites are numbered with the momentum of the states, we have the condition $k_1 + k_2 = k_3 + k_4$, which readily eliminates one of the sums in (3.22).

In the case of the geometry of a torus, we have periodic boundary conditions in the y -direction as well. Translating all the electrons the same number of lattice sites should not change the energy of the system. This leads to an extra requirement on k_i , i.e. we need only two indices on the V factor, and the Hamiltonian for the torus can be written

$$H = \sum_n \sum_{k>m} V_{km} c_{n+m}^\dagger c_{n+k}^\dagger c_{n+m+k} c_n. \quad (3.24)$$

The connection between V_{km} and $V_{k_1 k_2 k_3 k_4}$ can be understood to be:

$$\begin{aligned} V_{km} = & V_{n+m, n+k, n, n+m+k} - V_{n+m, n+k, n+m+k, n} \\ & - V_{n+k, n+m, n, n+m+k} + V_{n+k, n+m, n+m+k, n}. \end{aligned} \quad (3.25)$$

It is evident from eq. (3.24) that V_{k0} is the electrostatic interaction energy between two electrons at the distance k from each-other. V_{km} ($m \neq 0$) gives the energy of the hopping of two electrons at the distance $k+m$ changing to a distance of $k-m$, as illustrated by Fig. 3.4.

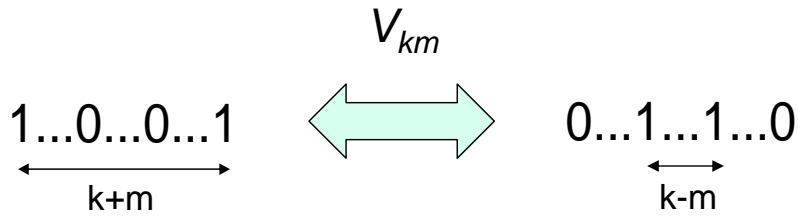


Figure 3.4: V_{km} is the matrix element for hopping between the two states above. Note that the hopping preserves the total momentum (or, equivalently, the position of the center of mass).

3.3 The thin limit

The idea of describing the quantum Hall system on a torus is of course a mathematical construction. When we let the circumferences in both the x - and the y -directions go to infinity, we will recover the real experimental planar geometry. It thus sounds a bit strange at first, wanting to explore the model in the opposite limit, i.e. letting $L_1 \rightarrow 0$. Surprisingly though, it has earlier been shown [7] that many of the physical features of the quantum Hall system survive in this limit. Moreover, the theoretical calculations and considerations become much simpler, why this limit should be profitable to explore. The reason for the simplification is the fact that, as $L_1 \rightarrow 0$, all hopping terms in the Hamiltonian vanish. The ground state (and other energy eigenstates) of the remaining electrostatic terms will just be certain "crystalline" configurations where the electrons are located in specific patterns that minimize the electrostatic repulsion.

3.3.1 What happens with the interaction terms

It can be seen from equation (3.23) that hopping (i.e. the quantum mechanical part of the Coulomb interaction) becomes unimportant as $L_1 \rightarrow 0$. As we already know, in this limit the lattice sites become widely separate. This means that the overlap between two wave functions centered at different lattice points tends to zero. For hopping terms, the wave functions for four different k -values are multiplied to give the matrix elements, thus they become extremely small. For the electrostatic exchange terms, the matrix elements are

$$V_{k_1 k_2 k_3 k_4} = \int \int \psi_{k_1}^*(\mathbf{r}_1) \psi_{k_2}^*(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2) \psi_{k_1}(\mathbf{r}_2) \psi_{k_2}(\mathbf{r}_1) d^2 r_1 d^2 r_2. \quad (3.26)$$

We see that only two different wave functions are involved. However, $\psi_{k_1}^*$, ψ_{k_1} and $\psi_{k_2}^*$, ψ_{k_2} are evaluated in different points in space, and hence even these overlaps will be extremely small. In contrast, the direct terms are

$$V_{k_1 k_2 k_3 k_4} = \int \int \psi_{k_1}^*(\mathbf{r}_1) \psi_{k_2}^*(\mathbf{r}_2) V(\mathbf{r}_1 - \mathbf{r}_2) \psi_{k_2}(\mathbf{r}_2) \psi_{k_1}(\mathbf{r}_1) d^2 r_1 d^2 r_2, \quad (3.27)$$

in which case we get non-zero overlaps even though ψ_{k_1} , ψ_{k_2} are widely separate.

We conclude that, in the thin limit, all interaction terms except for the direct electrostatic V_{k_0} are suppressed. The possibility to neglect all hopping and exchange terms of course simplifies many calculations.

3.3.2 Fractional charges in the thin limit

In section 2.3.1 we mentioned the exotic phenomenon of fractional charges in the quantum Hall system. We will now take advantage of the crystalline-like states in the thin limit to explore how fractional charges can be created in a lattice state [21]. We will argue that an electron/hole added to the $\nu = 1/q$ system gives rise to fractional charges $\pm e/q$ [21].

As an example, consider a lattice state at $\nu = 1/3$. In the thin limit we may consider the ground state

$$|\dots 100100100100100100100100100100\dots\rangle.$$

Now, what happens if we erase three holes at well separated positions, and at the same time add one unit cell 100 somewhere else⁸? The lattice state becomes

$$|\dots 100\underline{101}00100100\underline{101}00100100\underline{101}00100\dots\rangle.$$

At the three places where we erased a hole, there are concentrations of negative charge (red color and underlined in the sequence). But compared to the original lattice state above, there is only one more electron in total in the series. Therefore, we must see the three charge collections as sharing this extra charge $-e$. We conclude that each hole deleted from the $\nu = 1/3$ system has given rise to a fractional charge $-e/3$. Obviously, the opposite thing holds for adding holes, in which case they contribute with a charge $+e/3$ compared to the original system⁹. Note also that because the above example resulted in one extra electron charge, adding an electron to the $\nu = 1/3$ system yields three $-e/3$ charges. Adding one electron and one hole yields a ground state containing two $-e/3$ charges, and so on.

In general, adding one hole (i.e. a flux quantum) to the $\nu = 1/q$ system yields a contribution of charge $+e/q$. These fractional charges are actually the $L_1 \rightarrow 0$ limits of the quasiholes that appear in the bulk near the Laughlin fractions $\nu = \frac{1}{2m+1} = \frac{1}{q}$ for odd q . A construction similar to the above can be made to find the fractional charges of the Jain states at $\nu = \frac{p}{2mp+1}$, which are $\pm \frac{e}{2mp+1}$ [7]. Fractional charges appear in excitations of the ground states at

⁸In this way we can compare systems of the same size.

⁹Another way of looking at this is to simply add one hole to some $\nu = 1/3$ state, say $|100100100\rangle$, to yield $|1001001000\rangle$. The average charge density (total charge divided by the number of sites) for the $\nu = 1/3$ system is clearly $-e/3$. But the charge density at the site of the added hole is 0, and thus the change in the charge density at that position is $+e/3$.

the exact fillings. Here, one positive and one negative fractional charge of the same size appear at different places in the lattice states. Due to the opposite signs, the quasiparticles attract each-other and the more separate they are the larger energy has the excited state. However, when moving slightly away from the exact filling (e.g. adding or removing an empty site from $\nu = 1/3$), the fractional charges appear in the ground state.

The half-filled system is of particular interest to us. At $\nu = 1/2$, fractional charges appear only in the thin limit. For the state $|101010\dots\rangle$, adding an electron (without changing the number of sites) gives rise to two fractional charges $-e/2$, and adding a hole (site) gives $+e/2$. Adding one electron and one site yields one $-e/2$ charge. These formations are the crystalline ground states at and near half-filling in the thin limit. Since the bulk state $\nu = 1/2$ is gapless, fractional charges do not appear in that case. However, at $\nu = 5/2$, half-filling in the second Landau level, an interesting thing happens due to an additional non-trivial degeneracy of the ground state. Adding an electron plus one site (one site only) will, even in the bulk, lead to a ground state where the fractional charge $-e/2$ ($+e/2$) has divided into two charges of magnitude $e/4$. The formation of these fractional charges in the thin limit will be discussed in chapter 5.

3.3.3 Connection to the planar geometry

In recent studies of the QH system on the torus [7], the connection between the thin torus and the bulk has been explored. Good arguments have been given for that the crystal ground states (at $\nu = \frac{p}{2pm+1}$) in the thin limit evolve continuously to the ground states of the large system. For fillings with no gap (e.g. $\nu = 1/2$) on the other hand, there must be a phase transition between the two. However, this phase transition takes place on a very thin torus and the physics can still be understood by studying the physics on a thin (but not infinitely thin) torus. In this thesis, both half-fillings are treated. Especially the gapped states at filling fraction $\nu = 5/2$ are being considered, and the results again strongly indicate that there is a connection between the thin limit and the large system.

Chapter 4

Exploring half-filled Landau levels using pseudopotentials

4.1 Half-filling in the lowest Landau level

As previously hinted at, the fractional fillings without gaps (the metallic states) are as complicated to understand as the gapped ones. The absence of a gap is not only a consequence of the filling, but also of the electron-electron interaction. Hence, it is nontrivial to describe the metallic states. A successful wave function for the $\nu = \frac{1}{2m}$ ground state was constructed by Rezayi and Read [22] and looks like¹

$$\Psi_{\frac{1}{2m}}(z_1, z_2, \dots, z_N) = \mathcal{P}_{LLL} \det[e^{i\mathbf{k}_i \cdot \mathbf{r}_j}] \prod_{i < j} (z_i - z_j)^{2m} \prod_i e^{-\frac{1}{4}|z_i|^2}. \quad (4.1)$$

Remember the composite fermions from section 2.3 and the reduced magnetic field in eq. (2.8). With $\nu = 1/2$, $m = 1$, we get $B^* = B - B = 0$. We can thus picture this system as composite fermions moving in zero magnetic field—free composite fermions! From this perspective the Slater determinant of plane waves is logical. \mathcal{P}_{LLL} projects the state onto the lowest Landau level.

That $\nu = 1/2$ can be described by free composite fermions nicely explains why the state is gapless. The ground state is constructed by letting the free fermions fill the states with lowest momenta in a Fermi sea. The excitations are then realized by just deforming the Fermi sea—as the size of the system grows large, these excitation energies become extremely small since the Fermi sea becomes more dense.

Another microscopical explanation of the absence of a gap was given by Bergholtz and Karlhede [6], [7]. Their studies concerning fractional fillings in

¹Another model, a mean field theory, for these systems has been developed by Halperin, Lee and Read [23].

the lowest Landau level, on which this work is based, have been focused on the thin limit of the torus, and many interesting results have been obtained. For very thin tori, they find the ground state at $\nu = 1/2$ to be the crystal $|1010\dots\rangle$. At $L_1 \sim 5.3$, however, there is a phase transition to a homogeneous state. This state is described by an exact solution of a truncated Hamiltonian on a thin but finite torus. It then evolves continuously to the bulk system as $L_1 \rightarrow \infty$. The state gives a description of the half-filled system in terms of neutral dipoles in one dimension—a so-called Luttinger liquid. Since the net charge of the quasiparticles is zero, there is no coupling to the magnetic field and thus we are back at free particles. Thus, evidence was found that for $\nu = 1/2$, the infinitely thin torus and the bulk are separated by exactly one phase transition, whereas for fillings with a gap, the ground state of the infinitely thin torus evolves continuously to the bulk state². The main point is that in both cases the thin torus contains much of the physics of the experimental system. This will be of importance when we study the quantum Hall system on a torus using the method of pseudopotentials, which will be described in the following sections.

4.2 Half-filling in the second Landau level

The QH system at $\nu = 5/2$, i.e. the second Landau level being half-filled³, is one of the gapped systems. Hence, Jain’s picture of non-interacting composite fermions does not work here—one can not describe a state with a gap by free fermions. Another description is needed. In 1991, Moore and Read [26] wrote down a many-particle wave function that is believed to describe this system well. The function, which is also called the Pfaffian wave function, is a construction that describes the $\nu = 5/2$ system as being in the lowest Landau level. The completely filled energy level (lowest Landau level with both spin up and spin down) has all electrons in fixed states and they form a uniform background as seen by the other electrons. Hence they do not affect the interaction properties and may be disregarded. The Moore-Read wave function reads

$$\Psi_{\text{Pf}}(\{z_i\}) = \text{Pf} \left(\frac{1}{z_i - z_j} \right) \prod_{i < j} (z_i - z_j)^2 e^{-\frac{1}{4} \sum_i |z_i|^2}. \quad (4.2)$$

²The latter has been confirmed by analytical and numerical studies of the Laughlin wave function for $\nu = 1/3$ on the thin cylinder by Rezayi and Haldane [24] and by Seidel et al [25].

³In this thesis we will always refer to half-filling in the second Landau level as $\nu = 5/2$. $\nu = 1/2$ refers to half-filling in the lowest Landau level.

Note that this function is a product of the bosonic (symmetric) Laughlin wave function (lowest Landau level!) and an antisymmetric 'Pfaffian', $\text{Pf}\left(\frac{1}{z_i - z_j}\right)$, which makes the state fermionic. $\text{Pf}\left(\frac{1}{z_i - z_j}\right)$ is the square root of the determinant $\det\left(\frac{1}{z_i - z_j}\right)$, with all diagonal elements equal to zero. For example, for four particles it becomes $\text{Pf}\left(\frac{1}{z_i - z_j}\right) = \frac{1}{(z_1 - z_2)(z_3 - z_4)} - \frac{1}{(z_1 - z_3)(z_2 - z_4)} + \frac{1}{(z_1 - z_4)(z_2 - z_3)}$. The Pfaffian wave function is the exact ground state of a repulsive three-body interaction [27], [28] of the form

$$V(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k) = \mathcal{S}_{ijk} \nabla_i^4 \nabla_j^2 \delta^2(\mathbf{r}_i - \mathbf{r}_j) \delta^2(\mathbf{r}_j - \mathbf{r}_k), \quad (4.3)$$

where \mathcal{S}_{ijk} is a symmetrizer and all indices are summed over. This interaction prevents triples of electrons to be close to each-other.

In the real physical system, the electrons are of course interacting not via some strange three-body interaction, but via ordinary (two-body) Coulomb repulsion. In our lattice model we can construct the Hamiltonian of this system by calculating the matrix elements V_{km} using the single-particle wave functions in the second Landau level, ψ_{1k} , and the Coulomb interaction. Alternatively, we can describe the same physics⁴ by using the lowest Landau level wave functions ψ_{0k} but with a modified interaction, $V(r)$. For reasons that will become clear later, we choose the latter alternative in this thesis. There are advantages connected to studying the system in the lowest Landau level. One is that the single-particle wave functions⁵ of that level are holomorphic (i.e. analytical) and thus easier to handle.

Despite the similarity of half-filling, the quantum Hall systems at $\nu = 1/2$ and $\nu = 5/2$ are very different. A number of interesting and exotic features of the latter have been found by studying the Moore-Read wave function. It is possible to show that, if the system is studied in the geometry of a torus, the ground state of (4.2) is sixfold degenerate. Moreover, if one adds an extra electron to the 5/2-system without changing the number of single-particle states, the extra charge will split into four $-e/4$ charges located at different places. These fractional charges (quasiparticles) obey so-called non-Abelian statistics, as mentioned in section 2.3.1 and 3.3.2. We have seen earlier that the creation of fractional charges $\pm e/2$ in the thin limit of $\nu = 1/2$ is a trivial consequence of the crystalline formation. The appearance of $\pm e/4$ charges at $\nu = 5/2$, on the other hand, is a highly non-trivial phenomenon.

⁴I.e. obtain the same matrix elements V_{km} . This will yield the same energy spectrum and so on.

⁵This is true if we disregard a trivial gaussian factor common to all states. See eq. (3.10).

The Pfaffian wave function, and the exotic features of the states, have been found using conformal field theory (CFT). This is a field within physics that seemingly has a deep connection to the QHE but the precise reason for this is somewhat obscure. The fact is that the results we get from CFT provides a good description. An attempt to study the $\nu = 5/2$ system on the thin torus hopefully would lead to some understanding of the degenerate ground states and the origin of the fractional charges.

4.3 The pseudopotential method

The method of studying the quantum Hall system using so-called pseudopotentials was invented by Haldane in the 1980's [29].⁶ The idea is to make a kind of Taylor expansion of the potential and vary the expansion parameters. The Hamiltonian of any two-body interaction can be written in terms of pseudopotentials as

$$H = \sum_{i < j} \sum_{m=0}^{\infty} V_m P_m(M_{ij}), \quad (4.4)$$

where $P_m(M_{ij})$ projects onto a state where particles i, j have relative angular momentum m and V_m are the pseudopotential parameters, which are real numbers determined by the specific interaction. It can be shown that

$$V_m = \int_0^{\infty} q \tilde{V}(q) L_m(q^2) e^{-q^2} dq, \quad (4.5)$$

where $\tilde{V}(q) = \tilde{H}(q)$ is the Fourier transform of the potential⁷ and L_m are the Laguerre polynomials; $L_0(q^2) = 1$, $L_1(q^2) = 1 - q^2, \dots$

In principle, what we will do is to vary one or several of the pseudopotential parameters, i.e. let $V_m \rightarrow V_m + \delta V_m$, to simulate a new kind of interaction. Varying V_m will yield new matrix elements in our Hamiltonian, which in turn leads to new energy eigenstates corresponding to the new interaction. We will start from V_m as given by the ordinary Coulomb interaction and single-particle wave functions of the lowest Landau level, and vary them to yield new states corresponding to the Coulomb interaction in a higher Landau level.

The logical alternative to this, clearly, would be to keep the Coulomb interaction and just calculate the matrix elements from the single-particle wave

⁶Similar ideas have been elaborated by Trugman and Kivelson [30] and by Pokrovsky and Talapov [31]

⁷Remember that we disregard the kinetic energy since we study only one Landau level.

functions of the higher Landau level (see eq. (3.10) and (3.23)). However, varying the pseudopotential parameters gives an indication of how stable the various phases are, since it allows for continuous change of the interaction. Moreover, it offers the possibility to compare our results with similar studies [28].

In this thesis, we start from ordinary Coulomb interaction at $\nu = 1/2$ (half-filling in the lowest Landau level) and search for the energy eigenstates of $\nu = 5/2$ (half-filling in the second Landau level) with a gap.

Pseudopotentials in the lattice model

When we studied our lattice model earlier, we expressed the matrix elements $V_{k_1 k_2 k_3 k_4}$ in terms of the single-particle wave functions and the potential $V(\mathbf{r}_1 - \mathbf{r}_2)$ (see eq. (3.23)). In order to make a connection between these matrix elements and the pseudopotential parameters we must compute $V_{k_1 k_2 k_3 k_4}$ for a general interaction $V(\mathbf{r}_1 - \mathbf{r}_2)$ in the lowest Landau level. On the torus one finds that [32]

$$V_{k_1 k_2 k_3 k_4} = C \delta'_{k_1+k_2, k_3+k_4} \sum_{\mathbf{q}} \delta'_{k_1-k_4, q_2 L_2/2\pi} \tilde{V}(\mathbf{q}) e^{2\pi i q_1 (k_1-k_3)/L_2} e^{-q^2/2}, \quad (4.6)$$

where C is a constant, $\tilde{V}(\mathbf{q})$ is the two-dimensional Fourier transform of $V(\mathbf{r})$ and δ' is the periodic Kronecker delta (with period N_s). The sum is taken over all allowed wave vectors on the torus, i.e. $q_\alpha = \frac{2\pi n_\alpha}{L_\alpha}$, $n_\alpha = 0, \pm 1, \dots$

Now we use eq. (4.5) and the orthogonality of the Laguerre polynomials to relate a change in V_m to a change in $\tilde{V}(\mathbf{q}) = \tilde{V}(q) = \frac{1}{q}$ and thus in the matrix elements⁸. We have

$$\int_0^\infty 2q e^{-q^2} L_m(q^2) L_n(q^2) dq = \delta_{mn}, \quad (4.7)$$

which together with (4.5) yields

$$\tilde{V}(q) = 2 \sum_{m=0}^{\infty} V_m L_m(q^2). \quad (4.8)$$

Thus, in Fourier space, the change in the potential becomes

$$\delta \tilde{V} = 2 \sum_{m=0}^{\infty} \delta V_m L_m(q^2) \quad (4.9)$$

⁸For Coulomb, the Fourier transform of $V(\mathbf{r}) = \frac{1}{r}$ is $\tilde{V} = \frac{1}{q}$. A multiplicative factor $\frac{e^2}{\epsilon}$ has been contracted into the constant C in eq. (4.6). In all our computer simulations we use $\frac{e^2}{\epsilon} = 1$. Hence, in all plots in this thesis, δV_m are in units of $\frac{e^2}{\epsilon} = 1$.

and we vary our original matrix elements by letting

$$\tilde{V}(\mathbf{q}) = \frac{1}{q} \rightarrow \frac{1}{q} + \sum_{m=0}^{\infty} 2\delta V_m L_m(q^2) \quad (4.10)$$

in eq. (4.6). Here we use conventions such that $\delta V_m = 0, \forall m$ means Coulomb interaction in the lowest Landau level. For fermions, $\delta V_{2m} = 0$. Moreover, in this work we have also chosen to vary only δV_1 . In other words, we let

$$\tilde{V}(\mathbf{q}) \rightarrow \frac{1}{q} + 2\delta V_1(1 - q^2). \quad (4.11)$$

This is motivated by the fact that a short-range interaction is known to be very important in many quantum Hall systems⁹. According to previous studies, δV_3 gives a similar effect as δV_1 but with the opposite sign [28]. Moreover, the same studies have shown that it is sufficient to vary only δV_1 to find the Pfaffian phase.

Connection to real space interaction

We have seen how to change the interaction terms in the Hamiltonian by adding pseudopotential parameters to the Coulomb potential in Fourier space. But how do we relate the Laguerre polynomials $L_m(q^2)$ to a real interaction in position space, and can we get a simple picture of how they affect the matrix elements V_{km} ?

Let us first relate $\delta\tilde{V}(\mathbf{q}) = 2\delta V_1(1 - q^2)$ to a real space interaction. This is straightforward. We note that the Fourier transform of the delta function $\delta(\mathbf{r})$ is 1, since

$$\tilde{\delta}(\mathbf{q}) = \int e^{-i\mathbf{q}\cdot\mathbf{r}}\delta(\mathbf{r})d^2r = e^0 = 1. \quad (4.12)$$

Thus, writing

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^2} \int e^{i\mathbf{q}\cdot\mathbf{r}} d^2q \quad (4.13)$$

yields

$$\nabla^2\delta(\mathbf{r}) = \frac{1}{(2\pi)^2} \int -q^2 e^{i\mathbf{q}\cdot\mathbf{r}} d^2q, \quad (4.14)$$

i.e. the Fourier transform of $\nabla^2\delta(\mathbf{r})$ is $-q^2$. We conclude that the transformed potential $\delta\tilde{V}(\mathbf{q}) = 2\delta V_1(1 - q^2)$ corresponds to a real space interaction of the form

$$\delta V(\mathbf{r}) = 2\delta V_1(1 + \nabla^2)\delta(\mathbf{r}), \quad (4.15)$$

⁹The Laughlin wave functions are e.g. exact ground states when only the first expansion terms of (4.4) are non-vanishing.

where, again, δV_1 is the pseudopotential parameter we aim to vary.

So what is the physical interpretation of this? Since we are dealing with fermions, the delta function $\delta(\mathbf{r})$ will have no effect on the matrix elements—the electrons can not be at the same position. However, $\nabla^2\delta(\mathbf{r})$ gives rise to a positive potential energy for a pair of particles very close to each-other¹⁰, so adding a term of this kind to the Hamiltonian would favor states where the probability for electrons to come close is small. In contrast, a negative sign in front of $\nabla^2\delta(\mathbf{r})$ would lower the energy of pairs of closely lying electrons relative to the Coulomb case. For small factors δV_1 , however, the changes in the potential are small compared to the Coulomb interaction.

The phase we are searching for is the Pfaffian ground states of the 5/2 filling, which we know is the exact ground state of a repulsive three-body interaction that separates all triples of particles. One can easily realize that, written in the lattice language, the crystalline states that minimize this kind of interaction are of the type

$$|10101010\dots\rangle \tag{4.16}$$

and

$$|11001100\dots\rangle. \tag{4.17}$$

These are the Pfaffian lattice states in the thin limit. Remember that $|10101010\dots\rangle$ was what we expected as a ground state of the Coulomb interaction (i.e. for $\nu = 1/2$) in the thin limit. But $|11001100\dots\rangle$ is a new state that obviously is favored by a potential that allows for two electrons to be close to each-other. Our guess is therefore that we should choose negative values of δV_1 and thus add a negative $\nabla^2\delta(\mathbf{r})$ to the potential to find the $\nu = 5/2$ phase. Note that small values of δV_1 will let the repulsive Coulomb term remain important.

We have just argued that changing the potential with $\delta V(\mathbf{r}) = 2\delta V_1(1 + \nabla^2)\delta(\mathbf{r})$ for negative values of δV_1 will favor states where electrons come closer to each-other. In what way should this affect the interaction terms V_{km} ? From chapter 3 we remember that V_{k0} is the electrostatic interaction energy of two electrons separated k lattice constants. Favoring close-lying electrons will thus have the effect that V_{k0} for small k 's decrease in comparison to those for larger k 's when we turn on the pseudopotential parameters. One can show [33] that for $V(\mathbf{r}) = \nabla^2\delta(\mathbf{r})$,

$$V_{km} = (k^2 - m^2)e^{-2\pi^2(k^2+m^2)/L_1^2}. \tag{4.18}$$

¹⁰One can show by partial integration that the expectation value of $\nabla^2\delta(\mathbf{r})$ is nonzero. Although the wave function Ψ must go to zero for $\mathbf{r}_i = \mathbf{r}_j$, there is no such constraint on $\nabla\Psi$.

Hence, for small L_1 's, as we add this kind of potential, only small values of k and m will lead to any difference, while V_{km} for larger k 's will not be affected. For larger L_1 's, V_{k0} for larger k 's will be affected. This seems logical in the sense that the delta function is a pronounced short-range interaction. As L_1 increases, the lattice sites approach each-other, and electrons separated by more lattice distances get affected by the short-range interaction.

As a test of the program we have used to diagonalize the Hamiltonian we can check that this really happens. In Fig. 4.1 we see for $L_1 = 2$ how V_{10} decreases as we go to larger negative δV_1 , whereas the electrostatic terms for particles at larger distances increase. As we go to larger circumferences L_1 , not only V_{10} , but also V_{k0} for larger k 's decrease (see Fig. 4.2 and 4.3). In the computation all V_{k0} are shifted by an equal, unimportant, constant $\propto \delta V_1$. This leads to the increase of some V_{km} . However, the important thing is the relative magnitude of the matrix elements and one can ignore the 'apparent' change in V_{k0} for larger k -values.

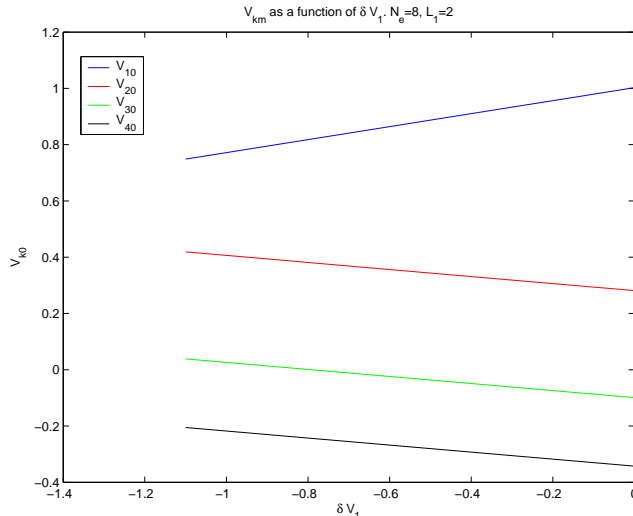


Figure 4.1: *The electrostatic interaction terms V_{k0} for $k = 1, 2, 3, 4$ as functions of the pseudopotential parameter δV_1 . Here, $L_1 = 2$. As we add the attractive short-distance interaction, V_{10} decreases in comparison to the other V_{k0} .*

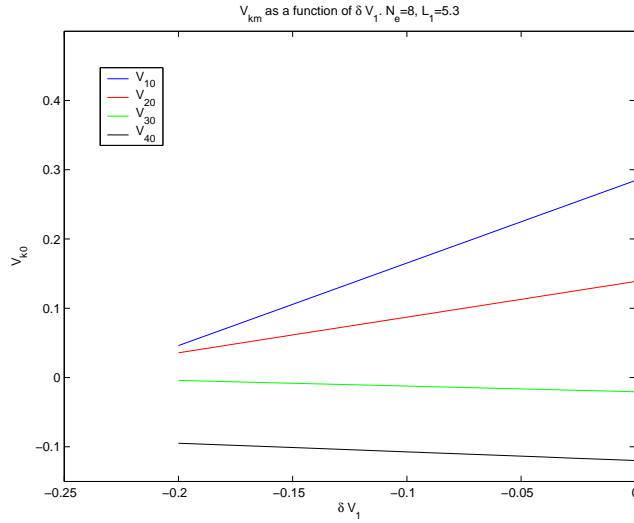


Figure 4.2: The electrostatic interaction terms V_{k0} for $k = 1, 2, 3, 4$ as functions of the pseudopotential parameter δV_1 . Here, $L_1 = 5.3$. As we add the attractive short-distance interaction, V_{10} and V_{20} decrease in comparison to the other V_{k0} . Note that V_{10} still is most affected.

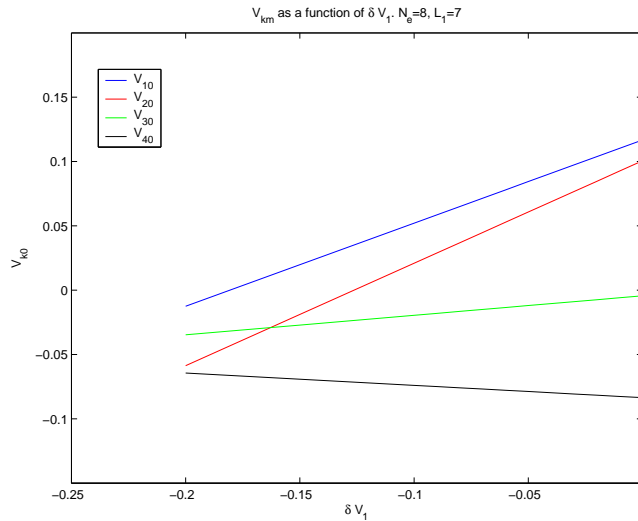


Figure 4.3: The electrostatic interaction terms V_{k0} for $k = 1, 2, 3, 4$ as functions of the pseudopotential parameter δV_1 . Here, $L_1 = 7$. As we add the attractive short-distance interaction, V_{10} , V_{20} and V_{30} decrease in comparison to V_{40} . Note that at this length, V_{20} is more affected than V_{10} and V_{30} .

Chapter 5

Our results

Using the pseudopotential method described in the previous chapter we have investigated the half-filled system on a torus. To find the energy eigenstates of our Hamiltonian we have used a computer program that performs exact numerical diagonalization¹. The input is the number of particles, N_e , and sites, N_s , the circumference L_1 and the potential (with or without added pseudopotential). The program computes the matrix elements of the Hamiltonian in the basis of the T_1 and T_2^q eigenstates and diagonalizes the matrix. This gives which superpositions of these eigenstates that are also eigenstates of H , and their energies. Each energy eigenstate is thus labeled by the energy and the quantum numbers K and N . The pseudopotential parameter δV_1 and the length L_1 have been varied to yield different ground states. Due to limited computer capacity we have been restricted to systems of up to ten particles. In particular we have focused on the case of $N_e = 8$.

5.1 A phase diagram for half-filling

As already mentioned, most of our computer simulations are done for $N_e = 8$, $N_s = 16$. In Fig. 5.1 we have illustrated the results of these in a phase diagram². The ground states of the various potentials and circumferences are separated by lines, and below we give a brief guide to the different phases.

For ordinary Coulomb interaction in the lowest Landau level (i.e. $\delta V_1 = 0$) we find the phases of $\nu = 1/2$. For L_1 up to ~ 5.3 , the ground state is the

¹This diagonalization program was written by Anders Karlhede and Emil Johansson Bergholtz.

²Phase diagrams are usually studied in thermodynamics, and we require, strictly speaking, the number of particles to be infinite. However, previous experience of analyzing quantum Hall systems has shown that even systems of very few particles provide a fairly accurate picture of the system in the thermodynamic limit.

state evolving from the crystalline state $|1010101010101010\rangle^3$. At $L_1 \sim 5.3$ there is a phase transition to the state described by the Luttinger liquid of neutral dipoles, as discussed in section 4.1. As L_1 increases even further, this evolves continuously to states with other quantum numbers but with the same physics [7]. We can read from the phase diagram that the ground states at $\nu = 1/2$ are quite stable against perturbations of the interaction, since they remain the ground states for a while when the potential is varied.

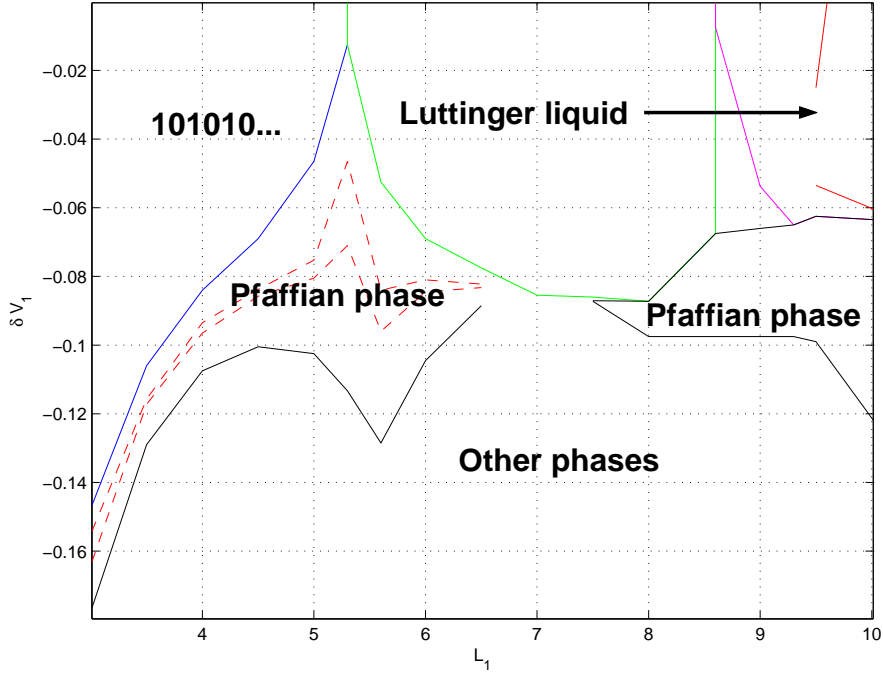


Figure 5.1: *Phase diagram for half-filling on the torus as a function of L_1 and δV_1 , eight particles. Along the L_1 -axis we find the ground states of the ordinary Coulomb interaction in the lowest Landau level. The crystalline state $|1010\dots\rangle$ is separated from the Luttinger liquid by a phase transition. The Luttinger state evolves continuously to the bulk state as L_1 increases. For $\delta V_1 \neq 0$ other ground states appear. Within the solid lines in the center of the diagram, we find the Pfaffian states. Within the dashed line these states are almost degenerate (their energy difference is less than 10% of the gap to the next state).*

As we decrease the pseudopotential parameter from zero we eventually get to a regime where the Pfaffian states are those with lowest energies—this area

³From now on we will disregard half of the states and just remember that there is a twofold translational degeneracy of every state—hence this example implies also 0101010101010101 .

is labeled 'Pfaffian' phase in Fig. 5.1. Within the dashed lines the Pfaffian states are even almost degenerate—their energies differ by less than 10% of the energy gap to the next state. One can note from the phase diagram that the Pfaffian states are rather stable in the sense that they allow for small changes in the potential (i.e. δV_1) and still remain the ground states. Also note that, as we make the torus thinner, it takes a larger δV_1 to get to the 5/2 phase. This can most probably be explained by eq. (4.18)—smaller L_1 needs a larger multiplicative factor to yield the same change in the potential.

As L_1 increases, one can follow the Pfaffian phase as it evolves to the larger system. Between approximately $L_1 = 7$ and $L_1 = 7.5$ we have not been able to find values of δV_1 where all the Pfaffian states are the lowest energy states. In this regime other states compete for these positions. However, at even larger circumferences the Pfaffian states take over again. It seems probable that it is possible to see the Pfaffian phase for all values of L_1 by adding other pseudopotential parameters than δV_1 and hence that it evolves continuously to the bulk state⁴. However, this remains to be investigated. From now on, we will only discuss what happens in the Pfaffian phase on the thin torus (specifically up to $L_1 \sim 7$).

The 'other phases' in the diagram are yet to be explored in detail and will not be further discussed in this thesis.

5.2 The Pfaffian phase

The first thing we need to clarify here is: How do we know that it is the Pfaffian phase we find when we decrease δV_1 like in Fig. 5.1? In section 4.3 we argued that the crystalline states that minimize a repulsive three-body interaction at half-filling are of the kind $|10101010\dots\rangle$ and $|11001100\dots\rangle$ when $L_1 \rightarrow 0$ (see eq. (4.16) and (4.17)). For $N_e = 8$ they are

$$\begin{aligned} &|1010101010101010\rangle; (K, N) = (8, 0), \\ &|0110011001100110\rangle + |1001100110011001\rangle; (K, N) = (4, 0), \\ &|0110011001100110\rangle - |1001100110011001\rangle; (K, N) = (4, 4), \end{aligned}$$

and the states obtained by acting with the translation operator T_2 on these states. This is of course the manifestation of the sixfold degeneracy of the Pfaffian ground state on the torus, and one can actually show that the quantum numbers above are those of the Pfaffian wave function [8]. For non-vanishing values of L_1 , the hopping terms are non-zero, hence the low energy

⁴Preliminary computer simulations for $N_e = 6$ indicate that this can be seen in this smaller system without varying more pseudopotential parameters.

states will be superpositions of many different lattice crystals. However, their symmetries and hence their quantum numbers remain unchanged. As argued earlier, this makes it profitable to first focus on the thin torus, where we can think of the crystalline states as 'parents' of the Pfaffian states in the bulk. In the regime labeled Pfaffian phase in the diagram, these are the ground states of the Hamiltonian.

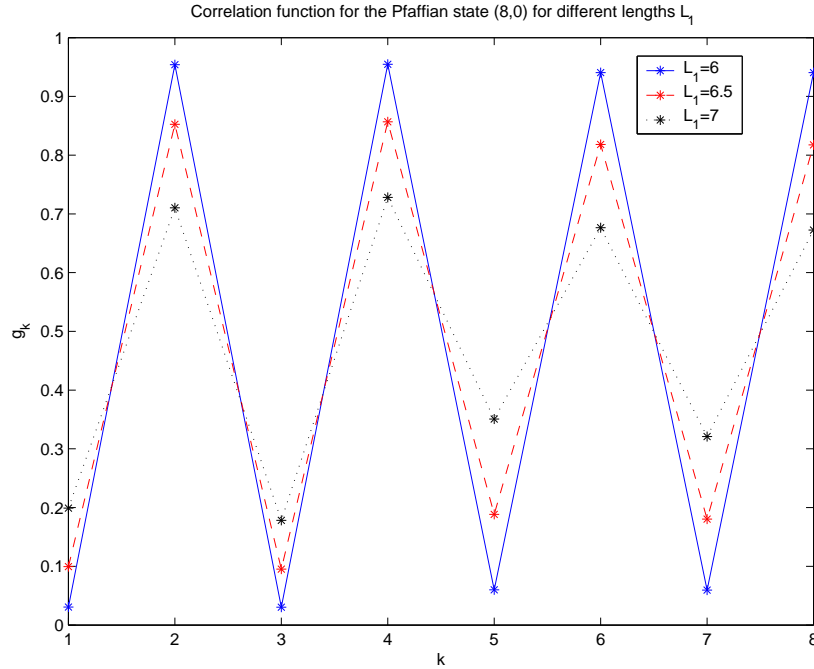


Figure 5.2: The correlation function g_k of the Pfaffian state $(8,0)$ plotted for different values of L_1 . The function evolves continuously from g_k for $|10101010101010\rangle$ in the thin limit. The lines between the points are just a guide to the eye.

Let us now investigate the properties of the Pfaffian states as expressed in the lattice language. The quantum numbers of the lowest energy eigenstates are given directly by the diagonalization. There are however many states with the same quantum numbers (K, N) but with different electron configurations. How can one check that $(8, 0)$, $(4, 0)$ and $(4, 4)$ really have the right structure? One way is to plot the average particle density at each site. This works fine for the state $(8, 0)$, since the particle density (10101010101010) is unique for the electron configuration $|10101010101010\rangle$. However, because of the construction of $(4, 0)$ and $(4, 4)$ above, they both (and many other possible states) are homogeneous, i.e. the particle density profile is

$(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})^5$. To be able to separate them from other homogeneous states, we study the correlation function

$$g_k = \frac{1}{N_e} \sum_{i=0}^{N_s-1} \langle n_i n_{i+k} \rangle. \quad (5.1)$$

Here, i labels the sites in the lattice and $n_i = 1$ for an occupied site and $n_i = 0$ for an empty. In words, the function g_k tells us the average number of neighbors at the distance k to the right of an electron. Consider e.g. the crystalline state $|10101010\dots\rangle$. Since every electron has one neighbor at the lattice distances $k = 2, 4, 6, \dots$ but none at distances $k = 1, 3, 5, \dots$, $g_k = 1$ for even k and $g_k = 0$ for odd k . In Fig. 5.2, one can see how the correlation function of the Pfaffian state which goes to $|10101010\dots\rangle$ in the thin limit, evolves as L_1 grows.

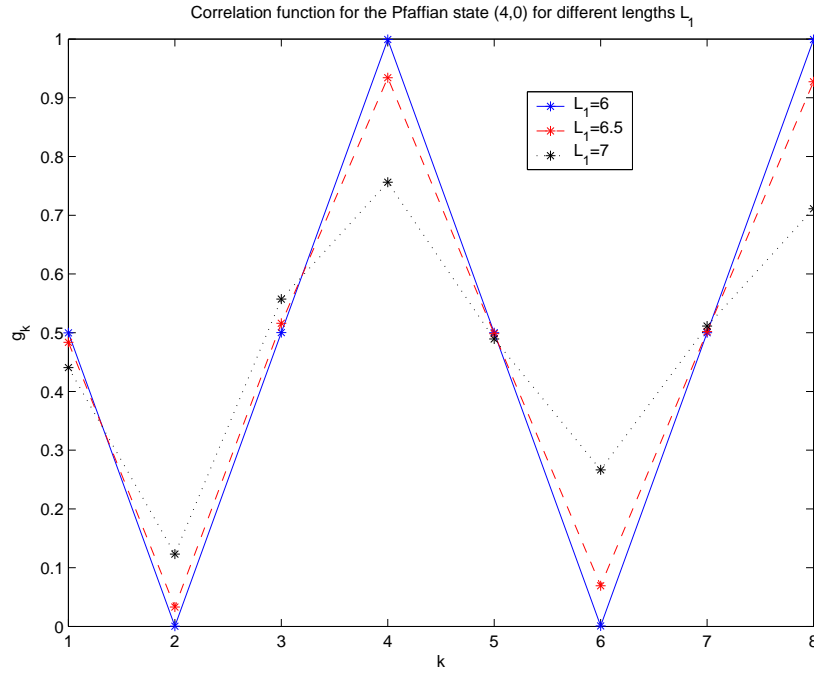


Figure 5.3: The correlation function g_k of the Pfaffian state $(4,0)$ plotted for different values of L_1 . The function evolves continuously from g_k for $|0110011001100110\rangle$ in the thin limit. The lines between the points are just a guide to the eye.

In the same manner, we look at the other Pfaffian states. One can easily realize that the correlation function for $|11001100\dots\rangle$ takes the values $g_k =$

⁵The normalized states are of course $\frac{1}{\sqrt{2}}(|0110011001100110\rangle \pm |1001100110011001\rangle)$.

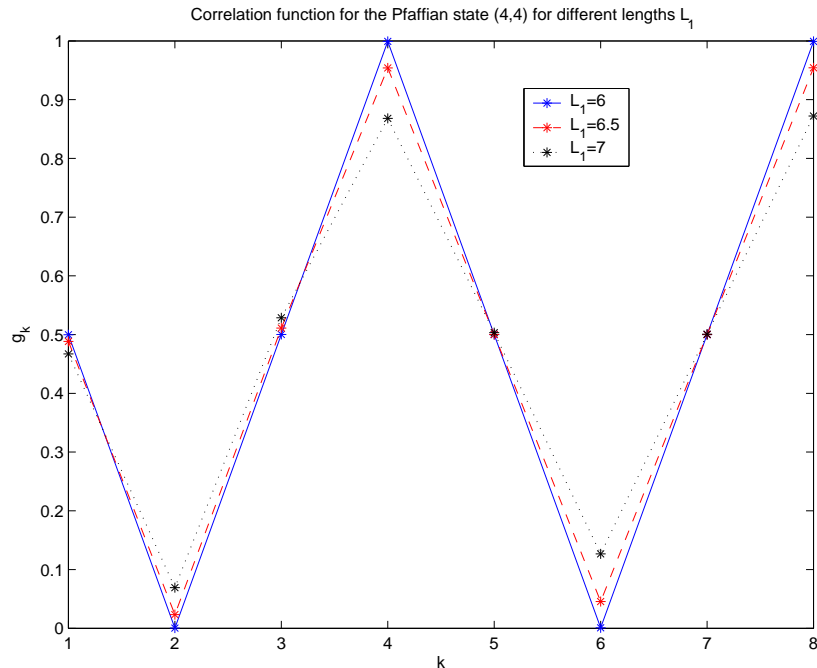


Figure 5.4: The correlation function g_k of the Pfaffian state $(4, 4)$ plotted for different values of L_1 . The function evolves continuously from g_k for $|0110011001100110\rangle$ in the thin limit. The lines between the points are just a guide to the eye.

$1/2$ for $k = 1, 3, 5, \dots$, $g_k = 0$ for $k = 2, 6, 10, \dots$ and $g_k = 1$ for $k = 4, 8, 12, \dots$. In Fig. 5.3 and 5.4 we have plotted g_k for the Pfaffian states with quantum numbers $(4, 0)$ and $(4, 4)$ for different values of L_1 .

Since the correlation functions of the states $(8, 0)$, $(4, 0)$ and $(4, 4)$ agree with the expected electron configurations, we conclude that they belong to the Pfaffian phase.

For lengths up to $L_1 = 7$ we see how the plotted correlations look similar and seem to change continuously.

Fractional charges

In section 3.3.2 we discussed how fractional charges form in a crystalline state. We concluded that adding a hole (site) to the $\nu = 1/2$ ground state in the thin limit changes the charge density at that position by $+e/2$. The question is: If we add one hole to the Pfaffian phase, will we be able to see how this extra charge density splits into two fractional charges $+e/4$? We have investigated this by looking at the ground states of $N_e = 8$, $N_s = 17$,

i.e. half-filling plus an extra hole, in the Pfaffian regime. Plotting the average particle density of the ground state gives a clear picture of the fractional charges especially in the thin limit (see Fig. 5.5). From the density profile,

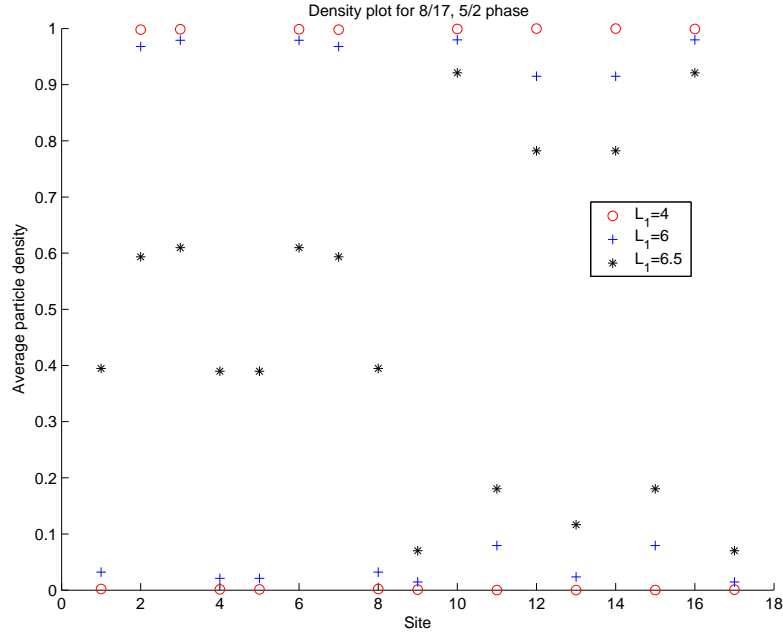


Figure 5.5: Plot of the average particle density at different sites. The filling fraction is $8/17$, i.e. we have added one hole to the half-filled system. The pseudopotential parameter is chosen to be in the Pfaffian regime. In the plot one can see how two fractional charges form in the domain walls between the two types of crystalline states. The density configuration seemingly evolves continuously as L_1 increases.

one can see that in this limit the state has the crystalline form

$$|0110011\underline{0010}101\underline{010}\rangle.$$

Clearly, the ground state at filling $8/17$ consists of alternating domains of the lattice configurations $|10101010\dots\rangle$ and $|11001100\dots\rangle$. This mixing is possible thanks to the very small difference in energy of these states in the Pfaffian regime (recall that it is already stated that degeneracy is required for the appearance of non-Abelian statistics).

At the interfaces, the so-called domain walls, between the two types of crystalline states, there are concentrations of positive charge (blue color and underlined in the sequence). The added hole with charge $e/2$ has split into two $e/4$ charges! In this way the fractional charges appear as domain walls

between the states $|10101010\dots\rangle$ and $|11001100\dots\rangle$. The interesting thing to note from the density plot is that this picture of the origin of the fractional charges seems to hold also for the thicker torus. Indeed, the change in the particle density of the ground state as L_1 increases indicates that it evolves continuously to the larger system (Fig. 5.5).

We have also searched for fractional charges in the low-lying spectrum at $N_e = 8$, $N_s = 16$ in the Pfaffian regime. One expects, since the positively and negatively charged quasiparticles attract each-other, that the lowest states will keep these charges close together while they will become more separate in higher excitations. Recall that the ground states in the thin limit are

$$\begin{aligned} &|1010101010101010\rangle; (K, N) = (8, 0), \\ &|0110011001100110\rangle + |1001100110011001\rangle; (K, N) = (4, 0), \\ &|0110011001100110\rangle - |1001100110011001\rangle; (K, N) = (4, 4). \end{aligned}$$

The computer simulations give that the first excited states are of the kind

$$|1010\underline{10110010}1010\rangle,$$

with a 32-fold degeneracy (16 T_2 translations times two because of a mirror symmetry). These states allows for two different interpretations. Either one may see it as the state $(8, 0)$ but with one electron moved one step (fractional charges $+e/2$ and $-e/2$ beside each-other). Or, one can see it as a string of $|1010\dots\rangle$ connected to a short sequence $|1100\rangle$ and two quarter charges very close to each-other in the domain walls (red and blue underlined). A similar situation holds for the second excited states, which look like

$$|0110011\underline{00101011}0\rangle$$

and have a 32-fold degeneracy. The difference is that now the background is of the kind $|11001100\dots\rangle$. In the third excited states, however, the quarter charged quasiparticles are more evident. These states,

$$|\underline{1011}0011\underline{0010}1010\rangle$$

(32-fold degeneracy) clearly consist of alternating strings $|1010\dots\rangle$ and $|1100\dots\rangle$. As expected, $-e/4$ (red underlined) and $+e/4$ (blue underlined) charges appear as domain walls.

Chapter 6

Conclusions and outlook

We have investigated the half-filled Landau level on the torus by varying the electron-electron interaction and the circumference, L_1 , of the torus.

We have verified that the Pfaffian state that is claimed to describe the $\nu = 5/2$ state is present also on the very thin torus. Here, the physics of the Pfaffian state becomes more transparent and we are able to get a simple understanding of phenomena that are otherwise only understood from conformal field theory, which has an unclear connection to the physical quantum Hall system. In particular, as $L_1 \rightarrow 0$ the hopping terms disappear and all energy eigenstates have fixed charges. The known sixfold degeneracy of the Pfaffian wave function is manifested as the six different crystals, 11001100 . . . , 10101010 . . . , and their translated versions. Moreover, moving slightly away from half-filling we find that the ground states emerge as alternating strings of these different vacua. The domain walls separating these have charge $\pm e/4$ which is exactly the charge of the quasiparticles as derived using CFT. Our numerics show that these $L_1 \rightarrow 0$ states develop continuously as L_1 increases, and thus can be viewed as 'parent' states of the bulk states. The fractional charges appear as the same kind of domain walls also in excited states at exactly half-filling.

Concerning the $\nu = 1/2$ system, we have verified that the ground state of this gapless system given earlier is stable against perturbations of the interaction.

Thinking about possible future research on this subject, there are several things that could be worth further investigation. One thing is to investigate what happens if we use ordinary Coulomb in the second Landau level instead of varying the pseudopotential parameters. Earlier work [28] has shown that this yields similar results in the bulk—the question is what happens as the torus becomes thin. Other things to do are e.g. to study systems of more particles and to make sure that the Pfaffian phase can be found using more

pseudopotential parameters for all L_1 . Another obvious thing to investigate is the 'other states' found in the phase diagram.

Summarizing, this and previous work has shown the great profits of studying the quantum Hall system on the thin torus, and there are presumably more interesting things to explore along this line.

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