Electrodynamics I – Extra problems

Extra problem no. 1

Consider a hollow sphere of radius R in vacuum. There is a surface charge, $\sigma = \sigma_0 \cos\theta$, on the sphere but no other charges are present. Compute the electrostatic potential and the electric field inside and outside of the sphere.

Extra problem no. 2

Start from the Biot-Sawart law,

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3} d^3x', \tag{1}$$

to show Gilberts law

$$\nabla \cdot \mathbf{B} = 0. \tag{2}$$

Then go on to show that the above, together with the continuity equation, leads to the Maxwell-Ampere law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{3}$$

Extra problem no. 3

Maxwell's equations in absence of sources ($\rho = 0$, $\mathbf{J} = \mathbf{0}$) have plane wave solutions,

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \qquad \mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}. \tag{4}$$

Show that these solutions gives the following constraint: $\mathbf{k} \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{B} = 0$.

Extra problem no. 4

Show, using Maxwell's equations, that the $\mathbf{E}$ and $\mathbf{B}$ fields obey the wave equation and that the electric charge is a conserved quantity.

Extra problem no. 5

Explain why we are allowed to choose gauges without changing the physics. Also explain the difference in arbitrariness of the gauge transformation between the Lorenz and the Coulomb gauge. (Hint: Think of the differences between the solutions of hyperbolic and elliptic differential equations.)

Extra problem no. 6

Derive the Lorentz-force from the action

$$\mathcal{S} = \int d\tau \{-m\sqrt{-\dot{x}_\alpha \dot{x}^\alpha} + e\dot{x}_\alpha A^\alpha(x(\tau))\}. \tag{5}$$

(Hint: You might find the problem easier by fixing $\tau = x^0 = t$ which gives the action $\mathcal{S} = \int dt\{-m\sqrt{1 - \dot{x}_i \dot{x}_i} + e\dot{x}_i A_i + eA_0\}$.)

Extra problem no. 7

Show that Maxwell's equations can be obtained from

$$\mathcal{S}[A] = -\int d^4x \left(\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + A_\alpha J^\alpha\right), \tag{6}$$

where $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and $J^\alpha(x)$ is a fixed external current. Also show that the action is invariant under the gauge transformation

$$A_\alpha(x) \to A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \Lambda(x), \quad J^\alpha \to J'^\alpha = J^\alpha \tag{7}$$

if J^α satisfies a simple (and familiar!) condition. What is the physical significance of this condition?

Extra problem no. 8

Derive the expression for length contraction from the constancy of the speed of light.

Extra problem no. 9

Consider two frames S and $\widetilde{S}$ with relative velocity v in the x_1 direction.

a) Obtain the electric and magnetic fields $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$ in terms of $\mathbf{E}$ and $\mathbf{B}$.

b) Place a charge q in frame S, at rest with coordinates x_i. Evaluate the electric and magnetic fields $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{B}}$ as measured by an observer in frame $\widetilde{S}$.

Extra problem no. 10

Consider a scalar field of the form

$$\phi(\mathbf{r}) = \frac{1}{r^2}, \qquad r^2 = x^2 + y^2 + z^2. \tag{8}$$

Perform a Lorentz boost in the t-x plane, and express the new function ϕ' as a function of the coordinates (t,x,y,z).